

Turán's problem for trees T_n with maximal degree $n - 4$

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Abstract

For $n \geq 6$ let $V = \{v_0, v_1, \dots, v_{n-1}\}$, $E_1 = \{v_0v_1, \dots, v_0v_{n-4}, v_1v_{n-3}, v_1v_{n-2}, v_1v_{n-1}\}$, $E_2 = \{v_0v_1, \dots, v_0v_{n-4}, v_1v_{n-3}, v_1v_{n-2}, v_2v_{n-1}\}$, $E_3 = \{v_0v_1, \dots, v_0v_{n-4}, v_1v_{n-3}, v_2v_{n-2}, v_3v_{n-1}\}$, $T_n^3 = (V, E_1)$, $T_n'' = (V, E_2)$ and $T_n''' = (V, E_3)$. In this paper, for $p \geq n \geq 15$ we obtain explicit formulas for $ex(p; T_n^3)$, $ex(p; T_n'')$ and $ex(p; T_n''')$, where $ex(p; L)$ denotes the maximal number of edges in a graph of order p not containing L as a subgraph.

MSC: Primary 05C35, Secondary 05C05

Keywords: tree, Turán problem

1. Introduction

In this paper, all graphs are simple graphs. For a graph $G = (V(G), E(G))$ let $e(G) = |E(G)|$ be the number of edges in G and let $\Delta(G)$ be the maximal degree of G . For a forbidden graph L , let $ex(p; L)$ denote the maximal number of edges in a graph of order p not containing L as a subgraph. The corresponding Turán's problem is to evaluate $ex(p; L)$.

Let \mathbb{N} be the set of positive integers, and let $p, n \in \mathbb{N}$ with $p \geq n \geq 3$. For a given tree T_n on n vertices, it is difficult to determine the value of $ex(p; T_n)$. The famous Erdős-Sós conjecture asserts that $ex(p; T_n) \leq \frac{(n-2)p}{2}$ for every tree T_n on n vertices. For the progress on the Erdős-Sós conjecture, see for example [2, 5]. Write $p = k(n-1) + r$, where $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. Let P_n be the path on n vertices. In [1] Faudree and Schelp showed that

$$(1.1) \quad ex(p; P_n) = k \binom{n-1}{2} + \binom{r}{2} = \frac{(n-2)p - r(n-1-r)}{2}.$$

Let $K_{1,n-1}$ denote the unique tree on n vertices with $\Delta(K_{1,n-1}) = n-1$, and for $n \geq 4$ let T_n' denote the unique tree on n vertices with $\Delta(T_n') = n-2$. In [3] the

¹The first author is supported by the National Natural Science Foundation of China (grant No. 11371163).

first author and Lin-Lin Wang determined $ex(p; K_{1,n-1})$ and $ex(p; T'_n)$. In [3,4] the first author and his coauthors also determined $ex(p; T_n)$ for trees T_n with n vertices and $\Delta(T_n) = n - 3$.

For $n \geq 6$ let

$$\begin{aligned} V &= \{v_0, v_1, \dots, v_{n-1}\}, \quad E_1 = \{v_0v_1, \dots, v_0v_{n-4}, v_1v_{n-3}, v_1v_{n-2}, v_1v_{n-1}\}, \\ E_2 &= \{v_0v_1, \dots, v_0v_{n-4}, v_1v_{n-3}, v_1v_{n-2}, v_2v_{n-1}\}, \\ E_3 &= \{v_0v_1, \dots, v_0v_{n-4}, v_1v_{n-3}, v_2v_{n-2}, v_3v_{n-1}\}. \end{aligned}$$

Suppose $T_n^3 = (V, E_1)$, $T_n'' = (V, E_2)$ and $T_n''' = (V, E_3)$. In this paper, for $p \geq n \geq 15$ we obtain explicit formulas for $ex(p; T_n^3)$, $ex(p; T_n'')$ and $ex(p; T_n''')$, see Theorems 3.1, 5.1 and 4.1-4.5.

In addition to the above notation, throughout this paper we also use the following notation: $[x]$ —the greatest integer not exceeding x , $d(v)$ —the degree of the vertex v in a graph, $d(u, v)$ —the distance between the two vertices u and v in a graph, K_n —the complete graph on n vertices, $K_{m,n}$ —the complete bipartite graph with m and n vertices in the bipartition, \overline{G} —the complement of G , $G[V_1]$ —the subgraph of G induced by vertices in the set V_1 , $G - V_1$ —the subgraph of G obtained by deleting vertices in V_1 and all edges incident with them, $\Gamma(v)$ —the set of vertices adjacent to the vertex v , $\Gamma_2(v)$ —the set of those vertices u such that $d(u, v) = 2$, $e(V_1V_1')$ —the number of edges with one endpoint in V_1 and another endpoint in V_1' .

2. Basic lemmas

Lemma 2.1. *Let $p, n \in \mathbb{N}$ with $p \geq n \geq 10$. Let T_n be a tree with n vertices and $\Delta(T_n) = n - 4$, and let $G \in Ex(p; T_n)$. Then $\Delta(G) \geq n - 5$.*

Proof. By [3, Theorem 2.1], $ex(p; K_{1,n-4}) = \lfloor \frac{(n-5)p}{2} \rfloor$. Since a graph does not contain $K_{1,n-4}$ as a subgraph implies that the graph does not contain any copies of T_n , we have

$$(2.1) \quad e(G) = ex(p; T_n) \geq ex(p; K_{1,n-4}) = \lfloor \frac{(n-5)p}{2} \rfloor.$$

If $\Delta(G) \leq n - 6$, using Euler's theorem we see that $e(G) = \frac{1}{2} \sum_{v \in V(G)} d(v) \leq \frac{(n-6)p}{2}$. Hence $\frac{(n-5)p-1}{2} \leq \lfloor \frac{(n-5)p}{2} \rfloor \leq e(G) \leq \frac{(n-6)p}{2}$. This is impossible. Thus $\Delta(G) \geq n - 5$.

Lemma 2.2. *Let $p, n \in \mathbb{N}$ with $p \geq n \geq 10$. Let T_n be a tree with n vertices and $G \in Ex(p; T_n)$. Suppose $V_1 \subset V(G)$ and $|V_1| = m + 1 \geq n - 3$. Then $e(G) - e(G - V_1) > 3m$.*

Proof. We first assume $m \geq n - 2$. Suppose $m + 1 = k(n - 1) + r$ with $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n - 2\}$. Then clearly $kK_{n-1} \cup K_r$ does not contain any copies of T_n and

$$\begin{aligned} e(kK_{n-1} \cup K_r) &= \frac{k(n-1)(n-2)}{2} + \frac{r(r-1)}{2} = \frac{(n-2)(m+1) - r(n-1-r)}{2} \\ &\geq \frac{(n-2)(m+1)}{2} - \frac{(n-1)^2}{8}. \end{aligned}$$

As $n \geq 10$ and $m+1 \geq n-1$ we have $(n-8)(m+1) \geq (n-8)(n-1) > \frac{(n-1)^2}{4} - 6$ and so

$$(2.2) \quad ex(m+1; T_n) \geq e(kK_{n-1} \cup K_r) \geq \frac{(n-2)(m+1)}{2} - \frac{(n-1)^2}{8} > 3m.$$

If $e(G) - e(G - V_1) \leq 3m$, then

$$e(G) < e(G - V_1) + e(kK_{n-1} \cup K_r) = e((G - V_1) \cup kK_{n-1} \cup K_r).$$

This contradicts the assumption $G \in Ex(p; T_n)$. If $m = n-3$ or $n-4$, then $e(K_{m+1}) = \frac{m(m+1)}{2} > 3m$. If $e(G) - e(G - V_1) \leq 3m$, then $e(G) < e(G - V_1) + e(K_{m+1}) = e((G - V_1) \cup K_{m+1})$, which contradicts the assumption $G \in Ex(p; T_n)$. Hence $e(G) - e(G - V_1) > 3m$ as claimed.

Lemma 2.3. *Let $p, n \in \mathbb{N}, p \geq n \geq 10$, $p = k(n-1) + r$, $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. Let T_n be a tree with n vertices, $\Delta(T_n) = n-4$ and $G \in Ex(p; T_n)$. If G is connected and $\Delta(G) \leq n-4$, then $p \leq \min\{\frac{3(n-1+r)+((-1)^n+(-1)^r)/2}{2}, \frac{r(n-1-r)}{2}\}$ and so $p \leq 2n-7$.*

Proof. Suppose that G is connected and $\Delta(G) \leq n-4$. By [3, Theorem 2.1], $ex(n-1+r; K_{1,n-4}) = \lfloor \frac{(n-1+r)(n-5)}{2} \rfloor$. Let $G_0 \in Ex(n-1+r; K_{1,n-4})$. Then G_0 does not contain T_n and so $(k-1)K_{n-1} \cup G_0$ does not contain T_n as a subgraph. Thus,

$$e((k-1)K_{n-1} \cup G_0) \leq ex(p; T_n) = e(G) \leq \left\lfloor \frac{(n-4)p}{2} \right\rfloor = \frac{(n-4)p - (1 - (-1)^{nr})/2}{2}.$$

On the other hand,

$$\begin{aligned} & e((k-1)K_{n-1} \cup G_0) \\ &= (k-1) \binom{n-1}{2} + \left\lfloor \frac{(n-1+r)(n-5)}{2} \right\rfloor \\ &= (k-1) \binom{n-1}{2} + \frac{(n-1+r)(n-5) - (1 - (-1)^{(n-1)(r-1)})/2}{2} \\ &= \frac{(n-4)p - (1 - (-1)^{nr})/2}{2} + p - \frac{3(n-1+r) + ((-1)^{nr} - (-1)^{(n-1)(r-1)})/2}{2} \\ &= \frac{(n-4)p - (1 - (-1)^{nr})/2}{2} + p - \frac{3(n-1+r) + ((-1)^n + (-1)^r)/2}{2}. \end{aligned}$$

Thus, $p \leq \frac{3(n-1+r)+((-1)^n+(-1)^r)/2}{2}$. We also have

$$\frac{(n-2)p - r(n-1-r)}{2} = e((k-1)K_{n-1} \cup K_r) \leq e(G) \leq \frac{(n-4)p}{2}$$

and so $p \leq \frac{r(n-1-r)}{2}$. Hence $p \leq \min\{\frac{3(n-1+r)+((-1)^n+(-1)^r)/2}{2}, \frac{r(n-1-r)}{2}\}$.

As $p \geq n$, we see that $r \notin \{0, 1, 2, n-3, n-2\}$ and so $p \leq \frac{3(n-1+n-4)+1}{2} = 3n-7$. If $p \geq 2(n-1)$, then $p = 2(n-1)+r$ with $0 \leq r \leq n-5$. As $2(n-1)+r > \frac{3(n-1+r)+1}{2}$, we get a contradiction. Hence $p < 2n-2$. Now we have $p = n-1+r$ with $0 \leq r \leq n-4$. As $\Delta(G) \leq n-4$ we have

$$ex(2n-5; T_n) = e(G) \leq \frac{(n-4)(2n-5)}{2} = n^2 - \frac{13}{2}n + 10 < n^2 - 6n + 11 = e(K_{n-1} \cup K_{n-4}),$$

which is a contradiction. Hence $p \leq 2n - 6$. As

$$ex(2n-6; T_n) = e(G) \leq \frac{(n-4)(2n-6)}{2} = n^2 - 7n + 12 < n^2 - 7n + 16 = e(K_{n-1} \cup K_{n-5}),$$

we get $p \leq 2n - 7$. This proves the lemma.

Lemma 2.4 ([4, Lemma 2.4]). *Let $n, n_1, n_2 \in \mathbb{N}$ with $n_1 < n - 1$ and $n_2 < n - 1$.*

- (i) *If $n_1 + n_2 < n$, then $\binom{n_1}{2} + \binom{n_2}{2} < \binom{n_1+n_2}{2}$.*
- (ii) *If $n_1 + n_2 \geq n$, then $\binom{n_1}{2} + \binom{n_2}{2} < \binom{n-1}{2} + \binom{n_1+n_2-n+1}{2}$.*

Lemma 2.5. *Let $n \in \mathbb{N}$ with $n \geq 10$, and let T_n be a tree with n vertices and $\Delta(T_n) = n - 4$. Suppose that for any positive integer $m \geq n$ and connected graph $H \in Ex(m; T_n)$ we have $\Delta(H) \leq n - 4$. Let $p \in \mathbb{N}$, $p \geq n$, $p = k(n - 1) + r$, where $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n - 2\}$. Assume that $G \in Ex(p; T_n)$, G is not connected, G_1, \dots, G_s are distinct components of G , $|V(G_i)| = p_i$ ($i = 1, 2, \dots, s$) and $p_1 \leq p_2 \leq \dots \leq p_s$. Then $p_1 \leq p_2 = \dots = p_{s-1} = n - 1 \leq p_s \leq 2n - 7$. If $p_1 < n - 1$ and $p_s \geq n$, then $p_1 \leq n - 7$ and $p_1(n - 3 - p_1) \leq p_1 + p_s + 1 \leq 2n - 7$.*

Proof. Suppose that $p_s \geq p_{s-1} \geq n$. Then clearly $G_{s-1} \in Ex(p_{s-1}; T_n)$, $G_s \in Ex(p_s; T_n)$ and $G_{s-1} \cup G_s \in Ex(p_{s-1} + p_s; T_n)$. By the assumption, $\Delta(G_{s-1}) \leq n - 4$ and $\Delta(G_s) \leq n - 4$. Hence,

$$ex(p_{s-1} + p_s; T_n) = e(G_{s-1} \cup G_s) \leq \frac{(n-4)(p_{s-1} + p_s)}{2}.$$

If $p_{s-1} + p_s < 3(n - 1) - 1$ and $G_0 \in Ex(p_{s-1} + p_s - (n - 1); K_{1, n-4})$, then K_{n-1} does not contain T_n and

$$\begin{aligned} e(K_{n-1} \cup G_0) &= \binom{n-1}{2} + \left[\frac{(n-5)(p_{s-1} + p_s - (n-1))}{2} \right] \\ &\geq \frac{(n-1)(n-2)}{2} + \frac{(n-5)(p_{s-1} + p_s - (n-1)) - 1}{2} \\ &= \frac{(n-5)(p_{s-1} + p_s) + 3(n-1) - 1}{2} \\ &> \frac{(n-4)(p_{s-1} + p_s)}{2} \geq ex(p_{s-1} + p_s; T_n). \end{aligned}$$

This contradicts the fact that $K_{n-1} \cup G_0$ does not contain T_n . Hence $p_{s-1} + p_s \geq 3(n - 1) - 1$. By Lemma 2.3, $p_{s-1} \leq 2n - 7$ and $p_s \leq 2n - 7$. Thus,

$$3(n - 1) - 1 \leq p_{s-1} + p_s \leq 2(2n - 7) < 6(n - 1) - 1.$$

Suppose $G_0 \in Ex(p_{s-1} + p_s - 2(n - 1); K_{1, n-4})$. Then G_0 does not contain T_n and $e(G_0) = \left[\frac{(n-5)(p_{s-1} + p_s - 2(n-1))}{2} \right]$. Thus,

$$\begin{aligned} &ex(p_{s-1} + p_s; T_n) \\ &\geq e(2K_{n-1} \cup G_0) = (n-1)(n-2) + \left[\frac{(n-5)(p_{s-1} + p_s - 2(n-1))}{2} \right] \\ &= 3(n-1) + \left[\frac{(n-5)(p_i + p_j)}{2} \right] \geq 3(n-1) + \frac{(n-5)(p_{s-1} + p_s) - 1}{2} \\ &= \frac{(n-4)(p_{s-1} + p_s) + 6(n-1) - 1 - (p_{s-1} + p_s)}{2} > \frac{(n-4)(p_{s-1} + p_s)}{2}. \end{aligned}$$

This is a contradiction.

By the above, $p_1 \leq p_2 \leq \dots \leq p_{s-1} \leq n-1$. We claim that $p_2 \geq n-1$. Otherwise, $p_1 \leq p_2 < n-1$ and $G_1 \cup G_2 \cong K_{p_1} \cup K_{p_2}$. If $p_1 + p_2 < n$, by Lemma 2.4(i) we have

$$e(G_1 \cup G_2) = e(K_{p_1} \cup K_{p_2}) = \binom{p_1}{2} + \binom{p_2}{2} < \binom{p_1 + p_2}{2} = e(K_{p_1 + p_2}).$$

Since $K_{p_1 + p_2}$ does not contain T_n and $G_1 \cup G_2 \in Ex(p_1 + p_2; T_n)$ we get a contradiction. Hence $p_1 + p_2 \geq n$. Using Lemma 2.4(ii) we see that

$$\begin{aligned} e(G_1 \cup G_2) &= e(K_{p_1} \cup K_{p_2}) = \binom{p_1}{2} + \binom{p_2}{2} \\ &< \binom{n-1}{2} + \binom{p_1 + p_2 - n + 1}{2} = e(K_{n-1} \cup K_{p_1 + p_2 - n + 1}). \end{aligned}$$

Since $p_1 \leq p_2 < n-1$, we have $p_1 + p_2 - n + 1 < n-1$. Hence $K_{n-1} \cup K_{p_1 + p_2 - n + 1}$ does not contain T_n . As $G_1 \cup G_2$ is an extremal graph without T_n , this is a contradiction. Thus, $p_2 \geq n-1$. Hence $p_1 \leq n-1 = p_2 = \dots = p_{s-1} \leq p_s \leq 2n-7$.

Assume that $p_s \geq n$ and $p_1 < n-1$. If $p_1 + p_s \geq 2n-5$, setting $G_0 \in Ex(p_1 + p_s - (n-1); K_{1, n-4})$ we find that G_0 does not contain T_n and $e(G_0) = \lfloor \frac{(n-5)(p_1 + p_s - (n-1))}{2} \rfloor$. Thus,

$$\begin{aligned} &e(K_{n-1} \cup G_0) \\ &= \binom{n-1}{2} + \lfloor \frac{(n-5)(p_1 + p_s - (n-1))}{2} \rfloor = \lfloor \frac{(n-5)(p_1 + p_s) + 3(n-1)}{2} \rfloor \\ &\geq \frac{(n-5)(p_1 + p_s) + 3(n-1) - 1}{2} \\ &= \frac{(n-4)p_s}{2} + \binom{p_1}{2} + \frac{3(n-1) - 1 + p_1(n-4-p_1) - p_s}{2} \\ &> \frac{(n-4)p_s}{2} + \binom{p_1}{2} \geq e(G_1 \cup G_s). \end{aligned}$$

This contradicts the fact $G_1 \cup G_s \in Ex(p_1 + p_s; T_n)$. Hence $p_1 + p_s \leq 2n-6$. If $p_1 \geq n-5$, then $p_s \leq 2n-6-p_1 \leq 2n-6-(n-5) = n-1$. This contradicts the assumption $p \geq n$. Hence $p_1 \leq n-6$. If $p_1 = n-6$, then $p_s = n$. As

$$\begin{aligned} e(K_{n-1} \cup K_{n-5}) &= \frac{(n-1)(n-2)}{2} + \frac{(n-5)(n-6)}{2} > \frac{(n-6)(n-7)}{2} + \frac{n(n-4)}{2} \\ &\geq e(G_1 \cup G_s) = ex(p_1 + p_s; T_n), \end{aligned}$$

we get a contradiction. Hence $p_1 \leq n-7$. We claim that $p_s \geq p_1(n-4-p_1)-1$. Otherwise, for $G_0 \in Ex(p_1 + p_s; K_{1, n-4})$ we have

$$\begin{aligned} e(G_0) &= \lfloor \frac{(n-5)(p_1 + p_s)}{2} \rfloor \geq \frac{(n-5)(p_1 + p_s) - 1}{2} \\ &> \frac{(n-4)p_s}{2} + \frac{p_1(p_1-1)}{2} \geq e(G_1 \cup G_s) = ex(p_1 + p_s; T_n), \end{aligned}$$

which is a contradiction. Hence the claim is true. As $p_1 + p_s \leq 2n - 6$, we get $p_1(n - 4 - p_1) - 1 \leq p_s \leq 2n - 6 - p_1$ and so $p_1(n - 3 - p_1) \leq p_1 + p_s + 1$. By Lemma 2.1, $\Delta(G_s) \leq n - 4$. Thus,

$$ex(p_1 + p_s; T_n) = e(G_1 \cup G_s) \leq \frac{p_1(p_1 - 1)}{2} + \frac{(n - 4)p_s}{2} = \frac{(p_1 + p_s)(n - 4) - p_1(n - 3 - p_1)}{2}.$$

On the other hand,

$$\begin{aligned} ex(p_1 + p_s; T_n) &\geq e(K_{n-1} \cup K_{p_1+p_s-(n-1)}) \\ &= \frac{(n-1)(n-2) + (p_1 + p_s - (n-1))(p_1 + p_s - n)}{2} \\ &= (n-1)^2 - \frac{(p_1 + p_s)(3n - 5 - p_1 - p_s) + (p_1 + p_s)(n - 4)}{2}. \end{aligned}$$

Hence $-p_1(n - 3 - p_1) \geq 2(n - 1)^2 - (p_1 + p_s)(3n - 5 - p_1 - p_s)$ and so

$$\begin{aligned} &(p_1 + p_s)(3n - 5 - p_1 - p_s) \\ &\geq 2(n - 1)^2 + p_1(n - 3 - p_1) \geq 2(n - 1)^2 + n - 4 = 2n^2 - 3n - 2. \end{aligned}$$

As $(2n - 6)(3n - 5 - (2n - 6)) = 2n^2 - 4n - 6 < 2n^2 - 3n - 2$ and $(2n - 7)(3n - 5 - (2n - 7)) = 2n^2 - 3n - 14 < 2n^2 - 3n - 2$, we get $p_1 + p_s \neq 2n - 6, 2n - 7$ and so $p_1 + p_s \leq 2n - 8$. Thus $p_1(n - 3 - p_1) \leq p_1 + p_s + 1 \leq 2n - 7$. This completes the proof.

Lemma 2.6. *Let $n \in \mathbb{N}$ with $n \geq 10$, and let T_n be a tree with n vertices and $\Delta(T_n) = n - 4$. Suppose that for any positive integer $m \geq n$ and connected graph $H \in Ex(m; T_n)$ we have $\Delta(H) \leq n - 4$. Let $p \in \mathbb{N}$ with $p \geq 2n - 6$. Then*

$$ex(p; T_n) = \frac{(n-1)(n-2)}{2} + ex(p - (n-1); T_n).$$

Proof. Let $G \in Ex(p; T_n)$. As $p \geq 2n - 6 > 2n - 7$, we see that G is not connected by Lemma 2.3. Suppose that G_1, \dots, G_s are all distinct components of G with $|V(G_i)| = p_i$ and $p_1 \leq p_2 \leq \dots \leq p_s$. Then clearly $G_i \in Ex(p_i; T_n)$ for $i = 1, 2, \dots, s$. By Lemma 2.5, $p_1 \leq p_2 = \dots = p_{s-1} = n - 1 \leq p_s \leq 2n - 7$. If $p_i = n - 1$ for some $i \in \{1, 2, \dots, s\}$, then clearly the result holds. If $p_i \neq n - 1$ for all $i = 1, 2, \dots$, then $s = 2$, $p_1 < n - 1 < n \leq p_2$. By Lemma 2.5, $p = p_1 + p_2 \leq 2n - 8$, which contradicts the assumption $p \geq 2n - 6$. Hence the theorem is proved.

Lemma 2.7. *Let $n \in \mathbb{N}$ with $n \geq 10$, and let T_n be a tree with n vertices and $\Delta(T_n) = n - 4$. Suppose that for any positive integer $m \geq n$ and connected graph $H \in Ex(m; T_n)$ we have $\Delta(H) \leq n - 4$. Assume $p, k \in \mathbb{N}$, $p = k(n - 1) + r$, $k \geq 2$ and $r \in \{0, 1, \dots, n - 2\}$. Then*

$$ex(p; T_n) = \frac{(n-2)(p - (n-1+r))}{2} + ex(n-1+r; T_n).$$

Proof. By Lemma 2.6,

$$\begin{aligned} &ex(p; T_n) \\ &= \sum_{s=2}^k (ex(s(n-1) + r; T_n) - ex((s-1)(n-1) + r; T_n)) + ex(n-1+r; T_n) \end{aligned}$$

$$= (k-1) \binom{n-1}{2} + \text{ex}(n-1+r; T_n).$$

Since $(k-1)(n-1) = p - (n-1+r)$ we deduce the result.

Lemma 2.8. *Let $n \in \mathbb{N}$, $n \geq 10$ and let T_n be a tree with n vertices and $\Delta(T_n) = n-4$. Suppose that for any positive integer $m \geq n$ and connected graph $H \in \text{Ex}(m; T_n)$ we have $\Delta(H) \leq n-4$. Assume $p \in \mathbb{N}$, $p = k(n-1) + r \geq n-1$, where $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. Then*

$$\frac{(n-2)p - r(n-1-r)}{2} \leq \text{ex}(p; T_n) \leq \frac{(n-2)p}{2} - \min \left\{ n-1+r, \frac{r(n-1-r)}{2} \right\}.$$

Hence, for $r \in \{0, 1, 2, n-5, n-4, n-3, n-2\}$ we have

$$\text{ex}(p; T_n) = \frac{(n-2)p - r(n-1-r)}{2}.$$

Proof. Since $kK_{n-1} \cup K_r$ does not contain T_n as a subgraph, we see that

$$\text{ex}(p; T_n) \geq e(kK_{n-1} \cup K_r) = \binom{n-1}{2} + \binom{r}{2} = \frac{(n-2)p - r(n-1-r)}{2}.$$

We claim that

$$\text{ex}(n-1+r; T_n) \leq \frac{(n-2)(n-1+r)}{2} - \min \left\{ n-1+r, \frac{r(n-1-r)}{2} \right\}.$$

As $\text{ex}(n-1; T_n) = e(K_{n-1}) = \binom{n-1}{2}$, we see that the claim holds for $r = 0$. Now suppose $r \geq 1$ and $G \in \text{Ex}(n-1+r; T_n)$. If G is connected, then $\Delta(G) \leq n-4$ and so $e(G) \leq \frac{(n-4)(n-1+r)}{2} = \frac{(n-2)(n-1+r)}{2} - (n-1+r)$. Thus the claim is true.

Now suppose that G is not connected and $G = G_1 \cup \dots \cup G_s$, where G_i is a component of G with $|V(G_i)| = p_i$ and $p_1 \leq p_2 \leq \dots \leq p_s$. By Lemma 2.5, $p_1 \leq p_2 = \dots = p_{s-1} = n-1 \leq p_s$. As $p_1 + \dots + p_s = n-1+r < 2(n-1)$ we see that $s = 2$, $p_1 < n-1$ and $p_2 = n-1+r-p_1 \geq n-1$. If $p_1 > r$, then clearly $p_2 < n-1$ and so $e(G) = e(K_{p_1} \cup K_{n-1+r-p_1}) = \binom{p_1}{2} + \binom{n-1+r-p_1}{2}$. Using Lemma 2.4(ii) we see that

$$e(G) = \binom{p_1}{2} + \binom{n-1+r-p_1}{2} < \binom{n-1}{2} + \binom{r}{2} = e(K_{n-1} \cup K_r).$$

This is a contradiction. Hence $p_1 \leq r$. If $p_1 < r$, then $p_2 = n-1+r-p_1 \geq n$ and so $\Delta(G) \leq n-4$. Using Lemma 2.5 we see that $p_1 \leq n-7$. Hence

$$\begin{aligned} e(G) &= e(G_1) + e(G_2) \leq \frac{p_1(p_1-1)}{2} + \frac{(n-4)(n-1+r-p_1)}{2} \\ &= \frac{(n-4)(n-1+r) - p_1(n-3-p_1)}{2} \\ &< \frac{(n-4)(n-1+r)}{2} = \frac{(n-2)(n-1+r)}{2} - (n-1+r). \end{aligned}$$

This shows that the claim is also true for $p_1 < r$. For $p_1 = r$ we see that

$$e(G) = e(K_{n-1} \cup K_r) = \frac{(n-1)(n-2) + r(r-1)}{2}$$

$$= \frac{(n-2)(n-1+r)}{2} - \frac{r(n-1-r)}{2}.$$

So the claim is also true. Hence the result is true for $p < 2n - 2$.

Now assume $p \geq 2n - 2$. By Lemma 2.7 and the above,

$$\begin{aligned} \text{ex}(p; T_n) &= \frac{(n-2)(p - (n-1+r))}{2} + \text{ex}(n-1+r; T_n) \\ &\leq \frac{(n-2)(p - (n-1+r))}{2} + \frac{(n-2)(n-1+r)}{2} \\ &\quad - \min \left\{ n-1+r, \frac{r(n-1-r)}{2} \right\} \\ &= \frac{(n-2)p}{2} - \min \left\{ n-1+r, \frac{r(n-1-r)}{2} \right\}. \end{aligned}$$

To complete the proof, we note that $\frac{r(n-1-r)}{2} \leq n-1-r$ for $r \in \{0, 1, 2, n-5, n-4, n-3, n-2\}$.

Lemma 2.9. *Let $n \in \mathbb{N}$, $n \geq 10$, $r \in \{0, 1, \dots, n-2\}$ and let T_n be a tree with n vertices and $\Delta(T_n) = n-4$. Suppose that for any positive integer $m \geq n$ and connected graph $H \in \text{Ex}(m; T_n)$ we have $\Delta(H) \leq n-5$. Then*

$$\text{ex}(n-1+r; T_n) = \max \left\{ \left\lceil \frac{(n-5)(n-1+r)}{2} \right\rceil, \binom{n-1}{2} + \binom{r}{2} \right\}.$$

Proof. Clearly $\text{ex}(n-1; T_n) = e(K_{n-1}) = \binom{n-1}{2}$. Thus the result is true for $r = 0$. Now assume $r \geq 1$. By [3, Theorem 2.1], $\text{ex}(n-1+r; K_{1, n-4}) = \left\lceil \frac{(n-5)(n-1+r)}{2} \right\rceil$. Since $\Delta(T_n) = n-4$ we see that $\text{ex}(n-1+r; T_n) \geq \text{ex}(n-1+r; K_{1, n-4}) = \left\lceil \frac{(n-5)(n-1+r)}{2} \right\rceil$. On the other hand, $\text{ex}(n-1+r; T_n) \geq e(K_{n-1} \cup K_r) = \binom{n-1}{2} + \binom{r}{2}$. Thus,

$$\text{ex}(n-1+r; T_n) \geq \max \left\{ \left\lceil \frac{(n-5)(n-1+r)}{2} \right\rceil, \binom{n-1}{2} + \binom{r}{2} \right\}.$$

Suppose $G \in \text{Ex}(n-1+r; T_n)$. If G is connected, then $\Delta(G) \leq n-5$ and so $e(G) \leq \left\lceil \frac{(n-5)(n-1+r)}{2} \right\rceil$. Hence

$$\begin{aligned} \text{ex}(n-1+r; T_n) = e(G) &\leq \left\lceil \frac{(n-5)(n-1+r)}{2} \right\rceil \\ &\leq \max \left\{ \left\lceil \frac{(n-5)(n-1+r)}{2} \right\rceil, \binom{n-1}{2} + \binom{r}{2} \right\}. \end{aligned}$$

This yields the result in this case.

Now suppose that G is not connected and $G = G_1 \cup \dots \cup G_s$, where G_i is a component of G with $|V(G_i)| = p_i$ and $p_1 \leq p_2 \leq \dots \leq p_s$. By the argument in the proof of Lemma 2.8, we have $s = 2$ and $p_1 \leq r$.

If $p_1 < r$, then $p_2 = n-1+r-p_1 \geq n$. Using Lemma 2.5 we see that $p_1 \leq n-7$. By the assumption, $\Delta(G_2) \leq n-5$ and so $e(G_2) \leq \left\lceil \frac{(n-5)p_2}{2} \right\rceil$. Hence

$$\begin{aligned} e(G) = e(G_1) + e(G_2) &\leq \frac{p_1(p_1-1)}{2} + \left\lceil \frac{(n-5)(n-1+r-p_1)}{2} \right\rceil \\ &= \left\lceil \frac{(n-5)(n-1+r) - p_1(n-4-p_1)}{2} \right\rceil \leq \left\lceil \frac{(n-5)(n-1+r) - 3p_1}{2} \right\rceil \end{aligned}$$

$$< \left\lceil \frac{(n-5)(n-1+r)}{2} \right\rceil.$$

This is a contradiction. Thus, $p_1 = r$ and so

$$e(G) = e(K_{n-1} \cup K_r) \leq \max \left\{ \left\lceil \frac{(n-5)(n-1+r)}{2} \right\rceil, \binom{n-1}{2} + \binom{r}{2} \right\}.$$

By the above, the lemma is proved.

Lemma 2.10. *Let $n \in \mathbb{N}$, $n \geq 10$ and let T_n be a tree with n vertices and $\Delta(T_n) = n - 4$. Suppose that for any positive integer $m \geq n$ and connected graph $H \in \text{Ex}(m; T_n)$ we have $\Delta(H) \leq n - 5$. Assume $p = k(n-1) + r \geq n - 1$, where $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. Then*

$$\text{ex}(p; T_n) = \frac{(n-2)p - r(n-1-r)}{2} + \max \left\{ 0, \left\lceil \frac{r(n-4-r) - 3(n-1)}{2} \right\rceil \right\}.$$

Proof. By Lemma 2.9,

$$\begin{aligned} & \text{ex}(n-1+r; T_n) \\ &= \max \left\{ \left\lceil \frac{(n-5)(n-1+r)}{2} \right\rceil, \binom{n-1}{2} + \binom{r}{2} \right\} \\ &= \frac{(n-1)(n-2) + r(r-1)}{2} + \max \left\{ 0, \left\lceil \frac{r(n-4-r) - 3(n-1)}{2} \right\rceil \right\} \\ &= \frac{(n-2)(n-1+r) - r(n-1-r)}{2} + \max \left\{ 0, \left\lceil \frac{r(n-4-r) - 3(n-1)}{2} \right\rceil \right\}. \end{aligned}$$

Thus the result is true for $p = n - 1 + r < 2n - 2$.

Now assume $p \geq 2n - 2$. From the above and Lemma 2.7 we see that

$$\begin{aligned} & \text{ex}(p; T_n) \\ &= \frac{(n-2)(p - (n-1+r))}{2} + \max \left\{ \left\lceil \frac{(n-5)(n-1+r)}{2} \right\rceil, \binom{n-1}{2} + \binom{r}{2} \right\} \\ &= \max \left\{ \frac{(n-2)p - r(n-1-r)}{2}, \left\lceil \frac{(n-2)p - 3(n-1+r)}{2} \right\rceil \right\} \\ &= \frac{(n-2)p - r(n-1-r)}{2} + \max \left\{ 0, \left\lceil \frac{r(n-4-r) - 3(n-1)}{2} \right\rceil \right\}. \end{aligned}$$

This completes the proof.

3. Evaluation of $\text{ex}(p; T_n'')$

Lemma 3.1. *Let $p, n \in \mathbb{N}$, $p \geq n \geq 10$ and $G \in \text{Ex}(p; T_n'')$. Suppose that G is connected. Then $\Delta(G) = n - 4$ or $n - 5$.*

Proof. By Lemma 2.1, $\Delta(G) \geq n - 5$. Thus it is sufficient to prove that $\Delta(G) \leq n - 4$. Suppose that $v_0 \in V(G)$, $d(v_0) = \Delta(G) = m$ and $\Gamma(v_0) = \{v_1, \dots, v_m\}$. If $p = m + 1$, then $V(G) = \{v_0, v_1, \dots, v_m\}$ and $m = p - 1 \geq n - 1$. Set $G' = G[v_1, \dots, v_m]$. If $d_{G'}(v_i) \geq 3$ for some $i = 1, 2, \dots, m$, as G does not contain T_n''

we see that $e(G') = d_{G'}(v_i) \leq m - 1$. Otherwise, we have $d_{G'}(v_i) \leq 2$ for every $i = 1, 2, \dots, m$ and so $e(G') \leq 2m/2 = m$. Hence we always have

$$e(G) = d(v_0) + e(G') \leq m + m = 2p - 2 < \frac{(n-5)p-1}{2} \leq \left\lceil \frac{(n-5)p}{2} \right\rceil.$$

This contradicts to (2.1). Thus $p > m + 1$.

Suppose that u_1, \dots, u_t are all vertices such that $d(u_1, v_0) = \dots = d(u_t, v_0) = 2$ and $\Gamma_2(v_0) = \{u_1, \dots, u_t\}$. Then $t \geq 1$. Assume $u_1 v_1 \in E(G)$ with no loss of generality. Set $V_1 = \{v_0, v_1, \dots, v_m\}$ and $V_2 = \{v_0, v_1, \dots, v_m, u_1\}$.

Suppose $t = 1$ and $m \geq n - 2$. If $d_{G'}(v_1) \geq 3$, as G does not contain T_n'' we see that $\{v_2, v_3, \dots, v_m\}$ is an independent set in G' . Hence $e(G) - e(G - V_1) \leq d(v_0) + d(u_1) + d(v_1) - 2 \leq 3m - 2$. If $d_{G'}(v_1) \leq 2$, as G does not contain T_n we see that $G[v_2, \dots, v_m]$ does not contain $K_{1,2}$. Set $G'' = G[v_2, \dots, v_m]$. Then $d_{G''}(v_i) \leq 1$ for $i = 2, 3, \dots, m$ and so $e(G'') = \frac{1}{2} \sum_{i=2}^m d_{G''}(v_i) \leq \frac{m-1}{2}$. Therefore,

$$e(G) - e(G - V_1) = e(G[V_1]) + d(u_1) \leq d(v_0) + 2 + e(G'') + d(u_1) \leq m + 2 + \frac{m-1}{2} + m < 3m.$$

From the above and Lemma 2.2 we see that $\Delta(G) \leq n - 3$ for $t = 1$.

Suppose $t = 1$ and $\Delta(G) = m \in \{n - 3, n - 4\}$. Then

$$\begin{aligned} e(G) - e(G - V_2) &\leq d(u_1) + e(G[v_0, v_1, \dots, v_m]) \\ &\leq m + e(K_{m+1}) = \frac{m^2 + 3m}{2} < \frac{(m+1)(m+2)}{2} = e(K_{m+2}). \end{aligned}$$

Thus, $e(G) < e((G - V_2) \cup K_{m+2})$, which contradicts the assumption $G \in Ex(p; T_n'')$.

By the above, for $t = 1$ we have $\Delta(G) \leq n - 5$. From now on we assume that $t \geq 2$. Suppose that $m = \Delta(G) \geq n - 3$, $|\Gamma(v_1) \cap \Gamma_2(v_0)| \geq 2$ and $u_1, u_2 \in \Gamma(v_1) \cap \Gamma_2(v_0)$. Then $\{v_2, v_3, \dots, v_m\}$ is an independent set. If $t = 2$ and $v_1 v_i \notin E(G)$ for $i = 2, 3, \dots, m$, then $e(G) - e(G - V_1) \leq d(v_0) + d(u_1) + d(u_2) \leq 3m$. If $t = 2$ and $v_1 v_i \in E(G)$ for some $i \in \{2, 3, \dots, m\}$, then $u_1 v_j, u_2 v_j \notin E(G)$ for $j \in \{2, 3, \dots, m\} - \{i\}$. Hence $e(G) - e(G - V_1) \leq d(v_1) + d(v_0) - 1 + 2 \leq m + m + 1$. For $t \geq 3$ we must have $u_3, \dots, u_t \in \Gamma(v_1)$ and $u_i v_j \notin E(G)$ for $i = 1, 2, \dots, t$ and $j = 2, 3, \dots, m$. Thus, $e(G) - e(G - V_1) \leq d(v_1) + d(v_0) - 1 \leq m + m - 1$. From the above we always have $e(G) - e(G - V_1) \leq 3m$. This contradicts Lemma 2.2. Hence $m = \Delta(G) \leq n - 4$ in the case $|\Gamma(v_1) \cap \Gamma_2(v_0)| \geq 2$.

Now suppose that $u_1 v_1, u_2 v_2, \dots, u_t v_t \in E(G)$ for $t \geq 2$. We first assume $m = \Delta(G) \geq n - 2$. If $t = 2$ and $v_1 v_2 \in E(G)$, then $d(v_3) = \dots = d(v_m) = 1$ and so

$$e(G) - e(G - V_1) \leq d(v_1) + d(v_2) - 1 + d(v_3) + \dots + d(v_m) \leq 4 + 3 + m - 2 = m + 5 < 3m.$$

If $t = 2$ and $v_1 v_2 \notin E(G)$, then clearly $d(v_i) \leq 3$ for $i = 1, 2, \dots, m$. Hence $e(G) - e(G - V_1) \leq d(v_1) + d(v_2) + \dots + d(v_m) \leq 3m$. For $t \geq 3$ we see that $d(v_i) \leq 2$ for $i = 1, 2, \dots, m$. Thus, $e(G) - e(G - V_1) \leq d(v_1) + \dots + d(v_m) \leq 2m$. From the above we always have $e(G) - e(G - V_1) \leq 3m$, which contradicts Lemma 2.2. Therefore $m = \Delta(G) \leq n - 3$.

Suppose $m = \Delta(G) = n - 3$. If $t = 2$, as G does not contain any copies of T_n'' we see that $v_i v_j \notin E(G)$ for $i \in \{1, 2\}$ and $j \in \{3, 4, \dots, n - 3\}$. Hence $e(G[v_0, v_1, \dots, v_{n-3}]) \leq e(K_{n-2}) - 2(n - 5)$. Thus,

$$e(G) - e(G - V_2) \leq d(u_1) + e(G[v_0, v_1, \dots, v_{n-3}]) + d(u_2)$$

$$\begin{aligned}
&\leq \binom{n-2}{2} - 2(n-5) + 2(n-3) = \frac{n^2 - 5n + 14}{2} \\
&< \frac{n^2 - 3n + 2}{2} = e(K_{n-1}).
\end{aligned}$$

This contradicts the fact $G \in \text{Ex}(p; T_n'')$. If $t \geq 3$, then $\Gamma(v_i) = \{v_0, u_i\}$ for $i = 1, 2, \dots, t$. Hence

$$\begin{aligned}
&e(G) - e(G - V_2) \\
&\leq d(u_1) + 1 + d(v_2) + \dots + d(v_t) + e(G[v_0, v_{t+1}, \dots, v_{n-3}]) + (t-1)(n-3-t) \\
&\leq n-3+1+2(t-1) + \frac{(n-2-t)(n-3-t)}{2} + (t-1)(n-3-t) \\
&= \frac{(n-1)(n-2) - (t^2 - 5t + 2n - 2)}{2}.
\end{aligned}$$

For $t \geq 3$ we have $t^2 - 5t + 2n - 2 \geq -6 + 2n - 2 > 0$ and so $e(G) - e(G - V_2) < \frac{(n-1)(n-2)}{2} = e(K_{n-1})$. That is, $e(G) < e(K_{n-1} \cup (G - V_1))$, which contradicts the assumption. Hence $\Delta(G) \leq n-4$ as claimed.

Lemma 3.2. *Let $p, n \in \mathbb{N}$, $p \geq n \geq 10$ and $G \in \text{Ex}(p; T_n'')$. Suppose that G is connected. Then $\Delta(G) = n-5$.*

Proof. By Lemma 3.1, $m = \Delta(G) \leq n-4$. Suppose that $v_0 \in V(G)$, $d(v_0) = \Delta(G) = n-4$, $\Gamma(v_0) = \{v_1, \dots, v_{n-4}\}$ and $\Gamma_2(v_0) = \{u_1, \dots, u_t\}$. By the proof of Lemma 3.1, we have $\Delta(G) = n-5$ for $t = 1$. From now on we assume $t \geq 2$. If $t = 2$ and $u_1v_1, u_2v_1 \in E(G)$, setting $V_1 = \{v_0, v_1, \dots, v_{n-4}\}$ and $V_2 = \{v_0, v_1, \dots, v_{n-4}, u_1, u_2\}$ we see that

$$\begin{aligned}
e(G) - e(G - V_2) &\leq d(u_1) + d(u_2) + e(G[V_1]) \\
&\leq n-4 + n-4 + \frac{(n-3)(n-4)}{2} = \frac{n^2 - 3n - 4}{2} \\
&< \frac{(n-1)(n-2)}{2} = e(K_{n-1})
\end{aligned}$$

and so $e(G) < e((G - V_2) \cup K_{n-1})$. This contradicts the assumption $G \in \text{Ex}(p; T_n'')$. If $t \geq 3$ and $v_1u_i \in E(G)$ for $i = 1, 2, \dots, t$, then $u_iv_j \notin E(G)$ for $i = 1, 2, \dots, t$ and $j = 2, 3, \dots, n-4$. Thus,

$$\begin{aligned}
e(G) - e(G - V_2) &\leq d(u_1) + d(u_2) + d(v_1) + e(G[v_0, v_2, v_3, \dots, v_{n-4}]) \\
&\leq n-4 + n-4 + n-4 + \frac{(n-4)(n-5)}{2} = \frac{n^2 - 3n - 4}{2} \\
&< \frac{(n-1)(n-2)}{2} = e(K_{n-1})
\end{aligned}$$

and so $e(G) < e((G - V_2) \cup K_{n-1})$, which contradicts the assumption $G \in \text{Ex}(p; T_n'')$.

Now suppose that $u_1v_1, \dots, u_tv_t \in E(G)$. Then $2 \leq t \leq n-4$. As $d(v_1) \leq n-4$ we see that $v_1v_i \notin E(G)$ for some $i \in \{2, 3, \dots, n-4\}$. Thus, for $t = 2$ we have

$$\begin{aligned}
&e(G) - e(G - V_2) \\
&\leq d(u_1) + d(u_2) + e(G[v_0, v_1, \dots, v_{n-4}])
\end{aligned}$$

$$\leq n - 4 + n - 4 + \binom{n-3}{2} - 1 = \frac{n^2 - 3n - 6}{2} < \binom{n-1}{2} = e(K_{n-1}).$$

This yields $e(G) < e((G - V_2) \cup K_{n-1})$, which is impossible. Hence $t \geq 3$.

Now suppose $u_1v_1, \dots, u_tv_t \in E(G)$, $t \geq 3$ and $V_1 = \{v_0, v_1, \dots, v_{n-4}, u_1\}$. We first claim that $d(v_i) \leq n - 5$ for $i = 1, 2, \dots, t$. Suppose $d(v_1) = n - 4$. Then $v_1v_i \notin E(G)$ for some $i \in \{2, 3, \dots, n-4\}$ and so $|\Gamma(u_1) \cap (G - V_1)| \leq 1$. Otherwise, for $w_1, w_2 \in \Gamma(u_1) \cap (G - V_1)$, $G[v_1, v_2, \dots, v_{n-4}, v_0, u_1, w_1, w_2]$ contains a copy of T_n'' . For $i \in \{1, 2, \dots, n-4\}$ there is at most one vertex in $\{u_1, \dots, u_t\}$ adjacent to v_i . Hence

$$\begin{aligned} e(G) - e(G - V_1) &\leq e(G[v_0, v_1, \dots, v_{n-4}]) + n - 4 + 1 \\ &\leq \binom{n-3}{2} - 1 + n - 4 + 1 = \frac{n^2 - 5n + 4}{2} < \binom{n-2}{2} = e(K_{n-2}) \end{aligned}$$

and so $e(G) < e((G - V_1) \cup K_{n-2})$. This is impossible. Hence the claim is true. Set $G' = G[v_0, v_1, \dots, v_{n-4}]$. Suppose that there are exactly s vertices v_{i_1}, \dots, v_{i_s} in $\{v_2, \dots, v_{n-4}\}$ adjacent to some vertex in $\{u_2, \dots, u_t\}$. Then $d(v_{i_j}) \leq n - 5$ for $j = 1, 2, \dots, s$ by the above argument. Hence

$$e(G') = \frac{1}{2} \sum_{i=0}^{n-4} d_{G'}(v_i) \leq \frac{(s+1)(n-6) + (n-4-s)(n-4)}{2} = \frac{n^2 - 7n + 10 - 2s}{2}$$

and therefore

$$\begin{aligned} e(G) - e(G - V_1) &\leq e(G') + d(u_1) + s \leq \frac{n^2 - 7n + 10}{2} - s + n - 4 + s \\ &= \frac{n^2 - 5n + 2}{2} < \frac{(n-2)(n-3)}{2} = e(K_{n-2}). \end{aligned}$$

This contradicts the fact $G \in Ex(p; T_n'')$. Thus $\Delta(G) = n - 5$ as claimed.

Theorem 3.1. *Let $p, n \in \mathbb{N}, p \geq n \geq 10$, $p = k(n-1) + r$, $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. Then*

$$ex(p; T_n'') = \frac{(n-2)p - r(n-1-r)}{2} + \max \left\{ 0, \left\lfloor \frac{r(n-4-r) - 3(n-1)}{2} \right\rfloor \right\}.$$

Proof. This is immediate from Lemmas 3.2 and 2.10.

4. Evaluation of $ex(p; T_n^3)$

Lemma 4.1. *Let $p, n \in \mathbb{N}, p \geq n \geq 10$ and $G \in Ex(p; T_n^3)$. Suppose that G is connected. Then $\Delta(G) = n - 5$ or $n - 4$.*

Proof. Suppose that $v_0 \in V(G), d(v_0) = \Delta(G) = m$ and $\Gamma(v_0) = \{v_1, \dots, v_m\}$. If $p = m + 1$, then $V(G) = \{v_0, v_1, \dots, v_m\}$ and $m = p - 1 \geq n - 4$. Since G does not contain T_n^3 , we see that $G[v_1, v_2, \dots, v_m]$ does not contain $K_{1,3}$ and hence $\Delta(G[v_1, v_2, \dots, v_m]) \leq 2$. Thus, $e(G[v_1, v_2, \dots, v_m]) \leq m$. Therefore

$$e(G) = d(v_0) + e(G[v_1, v_2, \dots, v_m]) \leq m + m = 2p - 2 < \left\lfloor \frac{(n-5)p}{2} \right\rfloor.$$

This contradicts to (2.1). Thus $p > m + 1$.

Suppose that $\Gamma_2(v_0) = \{u_1, \dots, u_t\}$. Then $t \geq 1$. We may suppose that v_1, \dots, v_{s_1} are all vertices adjacent to exactly two vertices in the set $\{u_1, \dots, u_t\}$ and $v_{s_1+1}, \dots, v_{s_2}$ are all vertices adjacent to exactly one vertex in the set $\{u_1, \dots, u_t\}$. Let $V_1 = \{v_0, v_1, \dots, v_m\}$, $V'_1 = V(G) - V_1$ and let $e(V_1 V'_1)$ be the number of edges with one endpoint in V_1 and another endpoint in V'_1 . As G does not contain T_n^3 we see that $e(V_1 V'_1) = 2s_1 + s_2 - s_1 = s_1 + s_2$.

If $m \geq n - 1$, as G does not contain T_n^3 as a subgraph, we see that $d(v_i) \leq 3$ for $i = 1, 2, \dots, m$ and so $e(G) \leq 3m + e(G - V_1)$. This contradicts Lemma 2.2. Hence $m \leq n - 2$.

Suppose $m = n - 2$. Since G does not contain T_n^3 we see that $d(v_i) \leq 3$ for $i = 1, 2, \dots, s_2$. Thus,

$$e(G) - e(G - V_1) \leq 3s_2 + \binom{n-1-s_2}{2} = s_2(s_2 - (2n-6)) + \binom{n-1}{2}.$$

As $s_2 \leq n - 2 < 2n - 6$ we have $e(G) < \binom{n-1}{2} + e(G - V_1) = e(K_{n-1} \cup (G - V_1))$. This contradicts the assumption $G \in Ex(p; T_n^3)$. Hence $m \leq n - 3$.

Suppose $m = n - 3$. For $i \in \{1, 2, \dots, s_1\}$ and $j \in \{s_1 + 1, \dots, n - 3\}$ we have $v_i v_j \notin E(G)$. Thus,

$$\begin{aligned} e(G) - e(G - V_1) &= n - 3 + 2s_1 + (s_2 - s_1) + e(G[v_{s_1+1}, \dots, v_{n-3}]) \\ &\leq n - 3 + 2s_1 + s_2 - s_1 + \binom{n-3-s_1}{2} \\ &= \binom{n-2}{2} + s_2 - \frac{s_1(2n-9-s_1)}{2}. \end{aligned}$$

If $s_1 \geq 2$, then $\frac{s_1(2n-9-s_1)}{2} \geq 2n - 11 > n - 3 \geq s_2$ and hence $e(G) < e(G - V_1) + \binom{n-2}{2} = e((G - V_1) \cup K_{n-2})$. This contradicts the fact $G \in Ex(p; T_n^3)$. Hence $s_1 = 0$ or 1 . We claim that $d(v_i) \geq n - 4$ for some $i \in \{1, 2, \dots, s_2\}$. Assume that $d(v_i) \leq n - 5$ for $i = 1, 2, \dots, s_2$. If $s_1 = 0$, then $e(G[V_1]) \leq \frac{s_2(n-6) + (n-2-s_2)(n-3)}{2} = \binom{n-2}{2} - \frac{3}{2}s_2$ and so

$$e(G) \leq \binom{n-2}{2} - \frac{3}{2}s_2 + s_2 + e(G - V_1) < e(K_{n-2}) + e(G - V_1) = e(K_{n-2} \cup (G - V_1)).$$

This contradicts the fact $G \in Ex(p; T_n^3)$. If $s_1 = 1$, we may assume that there are two vertices in $G - V_1$ adjacent to v_1 . Then $d(v_1) = 3$ and $v_1 v_i \notin E(G)$ for $i = 2, 3, \dots, n - 3$. Thus,

$$e(G[V_1]) \leq \frac{1 + (s_2 - 1)(n - 6) + (n - 2 - s_2)(n - 3)}{2} = \frac{(n - 2)(n - 3) - 3s_2 - (n - 7)}{2}$$

and so

$$\begin{aligned} e(G) &\leq e(G - V_1) + s_2 + 1 + \frac{(n - 2)(n - 3) - 3s_2 - (n - 7)}{2} \\ &= e(G - V_1) + \binom{n-2}{2} - \frac{s_2 + n - 9}{2} < e((G - V_1) \cup K_{n-2}). \end{aligned}$$

This contradicts the fact $G \in Ex(p; T_n^3)$. Hence the claim is true. Now suppose $d(v_1) \geq n - 4$ and $u_1 v_1 \in E(G)$, where $u_1 \in G - V_1$. Set $V_2 = \{u_1, v_0, v_1, \dots, v_{n-3}\}$ and $V_2' = V(G) - V_2$. Then there are at most two vertices in V_2' adjacent to u_1 . Suppose that there are exactly r vertices in $\{v_1, \dots, v_{s_2}\}$ adjacent to u_1 . As $s_1 = 0$ or 1 we have $e(V_2 V_2') \leq s_2 - r + 1 + 2$. As $\Delta(G) \leq n - 3$ we see that $d_{G[V_2]}(v_i) \leq n - 4$ for $i = 1, 2, \dots, s_2$. Thus,

$$e(G[V_2]) = r + e(G[V_1]) \leq r + \frac{s_2(n-4) + (n-2-s_2)(n-3)}{2} = \binom{n-2}{2} - \frac{s_2}{2} + r.$$

Therefore,

$$\begin{aligned} e(G) &= e(G[V_2]) + e(V_2 V_2') + e(G - V_2) \\ &\leq \binom{n-2}{2} - \frac{s_2}{2} + r + s_2 - r + 3 + e(G - V_2) = \binom{n-2}{2} + \frac{s_2 + 6}{2} + e(G - V_2) \\ &\leq \frac{(n-2)(n-3) + n - 3 + 6}{2} + e(G - V_2) \\ &< \binom{n-1}{2} + e(G - V_2) = e(K_{n-1} \cup (G - V_2)), \end{aligned}$$

which contradicts the assumption $G \in Ex(p; T_n^3)$. Hence $\Delta(G) \leq n - 4$.

Theorem 4.1. *Let $p, n \in \mathbb{N}$, $p \geq n \geq 10$, $p = k(n - 1) + r$, $k \in \mathbb{N}$ and $r \in \{0, 1, 2, n - 5, n - 4, n - 3, n - 2\}$. Then*

$$ex(p; T_n^3) = \frac{(n-2)p - r(n-1-r)}{2}.$$

Proof. This is immediate from Lemmas 4.1 and 2.10.

Lemma 4.2. *Let $n \in \mathbb{N}$, $n \geq 10$, $p = n - 1 + r$, $r \in \{1, 2, \dots, n - 6\}$ and $G \in Ex(p; T_n^3)$. Suppose that G is connected. If $r(n - 8 - r) > 5 + ((-1)^n - (-1)^{(n-1)(r-1)})/2$, then $\Delta(G) = n - 5$.*

Proof. By Lemma 4.1, $\Delta(G) = n - 4$ or $n - 5$. Suppose $\Delta(G) = n - 4$, $v_0 \in V(G)$, $d(v_0) = n - 4$, $\Gamma(v_0) = \{v_1, \dots, v_{n-4}\}$ and $\Gamma_2(v_0) = \{u_1, \dots, u_t\}$. Then $1 \leq t \leq n - 4$. We may suppose that v_1, \dots, v_{s_1} are all vertices adjacent to exactly two vertices in $\Gamma_2(v_0)$ and $v_{s_1+1}, \dots, v_{s_2}$ are all vertices adjacent to exactly one vertex in $\Gamma_2(v_0)$.

We first claim that $d(v_i) \leq n - 5$ for $i = 1, 2, \dots, s_1$. Suppose $d(v_1) = n - 4$ and u_1, u_2 are two vertices in $G - \{v_0, v_1, \dots, v_{n-4}\}$ adjacent to v_1 . Set $G' = G[v_0, v_1, \dots, v_{n-4}]$. Then

$$\begin{aligned} e(G') &= \frac{1}{2} (d_{G'}(v_0) + \sum_{i=1}^{s_1} d_{G'}(v_i) + \sum_{i=s_1+1}^{s_2} d_{G'}(v_i) + \sum_{i=s_2+1}^{n-4} d_{G'}(v_i)) \\ &\leq \frac{1}{2} (n - 4 + (n - 6)s_1 + (n - 5)(s_2 - s_1) + (n - 4 - s_2)(n - 4)) \\ &= \frac{n^2 - 7n + 12 - s_1 - s_2}{2} \end{aligned}$$

Let $V_1 = \{v_0, v_1, \dots, v_{n-4}, u_1, u_2\}$ and $V_1' = V(G) - V_1$. As $d(v_1) = n - 4$, for $i = 1, 2$ there are at most two vertices in $G - V_1$ adjacent to u_i . Thus,

$$e(G) - e(G - V_1) = e(G[V_1]) + e(V_1 V_1') \leq e(G') + 2s_1 + s_2 - s_1 + 2 + 2$$

$$\begin{aligned}
&\leq \frac{n^2 - 7n + 12 - s_1 - s_2}{2} + s_1 + s_2 + 4 = \frac{n^2 - 7n + 20 + s_1 + s_2}{2} \\
&\leq \frac{n^2 - 7n + 20 + n - 4 + n - 4}{2} = \frac{n^2 - 5n + 12}{2} \\
&< \frac{(n-1)(n-2)}{2} = e(K_{n-1}).
\end{aligned}$$

Hence $ex(p; T_n^3) = e(G) < e(G - V_1) + e(K_{n-1}) = e((G - V_1) \cup K_{n-1})$. As $(G - V_1) \cup K_{n-1}$ does not contain T_n^3 , we get a contradiction. Hence $d(v_i) \leq n - 5$ for $i = 1, 2, \dots, s_1$.

Now we show that $d(v_i) \leq n - 5$ for $s_1 < i \leq s_2$. Suppose $d(v_i) = n - 4$ for $i \in \{s_1 + 1, \dots, s_2\}$ and $u_i v_i \in E(G)$, where $u_i \in V(G) - \{v_0, v_1, \dots, v_{n-4}\}$. For $G' = G[v_0, v_1, \dots, v_{n-4}]$, from the above and the fact $\Delta(G) \leq n - 4$ we see that $d_{G'}(v_i) \leq n - 7$ for $1 \leq i \leq s_1$, $d_{G'}(v_i) \leq n - 5$ for $s_1 < i \leq s_2$ and $d_{G'}(v_i) \leq n - 4$ for $s_2 < i \leq n - 4$. Thus

$$\begin{aligned}
e(G') &= \frac{1}{2} \sum_{i=0}^{n-4} d_{G'}(v_i) \\
&\leq \frac{1}{2} (n - 4 + (n - 7)s_1 + (n - 5)(s_2 - s_1) + (n - 4 - s_2)(n - 4)) \\
&= \frac{n^2 - 7n + 12 - 2s_1 - s_2}{2}.
\end{aligned}$$

Set $V_1 = \{v_0, v_1, \dots, v_{n-4}, u_1\}$ and $V'_1 = V(G) - V_1$. As $d(v_1) = n - 4$, there are at most two vertices in $G - V_1$ adjacent to u_1 . Note that $s_2 \leq n - 4$. We deduce that

$$\begin{aligned}
e(G) - e(G - V_1) &= e(G[V_1]) + e(V_1 V'_1) \\
&\leq \frac{n^2 - 7n + 12 - 2s_1 - s_2}{2} + 2s_1 + s_2 - s_1 + 2 = \frac{n^2 - 7n + 16 + s_2}{2} \\
&\leq \frac{n^2 - 7n + 16 + n - 4}{2} = \frac{n^2 - 6n + 12}{2} < \frac{(n-2)(n-3)}{2} = e(K_{n-2}).
\end{aligned}$$

Hence $e(G) < e((G - V_1) \cup K_{n-2})$. This contradicts the assumption $G \in Ex(p; T_n^3)$.

By the above,

$$(4.1) \quad d(v_i) \leq n - 5 \quad \text{for } i = 1, 2, \dots, s_2.$$

For $V_1 = \{v_0, v_1, \dots, v_{n-4}\}$ we have $|V(G - V_1)| = p - (n - 3) = r + 2 < n$ and so $e(G - V_1) \leq \binom{r+2}{2}$. As $\Delta(G) \leq n - 4$ and $d_G(v_i) \leq n - 5$ for $i = 1, 2, \dots, s_2$, we see that $d_{G[V_1]}(v_i) \leq n - 7$ for $1 \leq i \leq s_1$, $d_{G[V_1]}(v_i) \leq n - 6$ for $s_1 < i \leq s_2$ and $d_{G[V_1]}(v_i) \leq n - 4$ for $s_2 < i \leq n - 4$. Thus,

$$\begin{aligned}
e(G[V_1]) &= \frac{1}{2} \sum_{i=0}^{n-4} d_{G[V_1]}(v_i) \\
(4.2) \quad &\leq \frac{1}{2} (n - 4 + (n - 7)s_1 + (n - 6)(s_2 - s_1) + (n - 4)(n - 4 - s_2)) \\
&= \frac{n^2 - 7n + 12 - s_1 - 2s_2}{2}.
\end{aligned}$$

Set $V'_1 = V(G) - V_1$. Then

$$\begin{aligned}
e(G) &= e(G[V_1]) + e(V_1 V'_1) + e(G - V_1) \\
&\leq \frac{n^2 - 7n + 12 - s_1 - 2s_2}{2} + 2s_1 + s_2 - s_1 + \binom{r+2}{2} \\
&= \frac{n^2 - 7n + 12 + s_1 + (r+1)(r+2)}{2} \\
&\leq \frac{n^2 - 7n + 12 + n - 4 + r^2 + 3r + 2}{2} = \frac{n^2 - 6n + 10 + r^2 + 3r}{2}
\end{aligned}$$

and so

$$e(G) \leq \left\lceil \frac{n^2 - 6n + 10 + r^2 + 3r}{2} \right\rceil = \frac{n^2 - 6n + 10 + r^2 + 3r - (1 - (-1)^n)/2}{2}.$$

Suppose $G_0 \in \text{Ex}(n-1+r; K_{1,n-4})$. Then

$$e(G_0) = \left\lceil \frac{(n-1+r)(n-5)}{2} \right\rceil = \frac{(n-1+r)(n-5) - (1 - (-1)^{(n-1)(r-1)})/2}{2}.$$

As G_0 does not contain T_n^3 and $G \in \text{Ex}(n-1+r; T_n^3)$, we get

$$\begin{aligned}
&\frac{n^2 - 6n + 10 + r^2 + 3r - (1 - (-1)^n)/2}{2} \\
&\geq e(G) \geq e(G_0) \geq \frac{(n-1+r)(n-5) - (1 - (-1)^{(n-1)(r-1)})/2}{2}
\end{aligned}$$

and so $r(n-8-r) \leq 5 + ((-1)^n - (-1)^{(n-1)(r-1)})/2$. Hence, if $r(n-8-r) > 5 + ((-1)^n - (-1)^{(n-1)(r-1)})/2$, we must have $\Delta(G) < n-4$ and so $\Delta(G) = n-5$ as claimed.

Lemma 4.3. *Let $n, r \in \mathbb{N}$, $n \geq 15$ and $3 \leq r \leq n-9$. Then*

$$\text{ex}(n-1+r; T_n^3) = \max \left\{ \left\lceil \frac{(n-5)(n-1+r)}{2} \right\rceil, \binom{n-1}{2} + \binom{r}{2} \right\}.$$

Proof. By the proof of Lemma 2.9, we have

$$\text{ex}(n-1+r; T_n^3) \geq \max \left\{ \left\lceil \frac{(n-5)(n-1+r)}{2} \right\rceil, \binom{n-1}{2} + \binom{r}{2} \right\}.$$

For $3 \leq r \leq n-10$ we see that $r(n-8-r) \geq 2(n-10) > 7 > 5 + ((-1)^n - (-1)^{(n-1)(r-1)})/2$. For $r = n-9$ we also have $r(n-8-r) = n-9 > 5 + ((-1)^n - (-1)^{(n-1)(r-1)})/2$. Let $G \in \text{Ex}(n-1+r; T_n^3)$. If G is connected, by Lemma 4.2 we have $\Delta(G) \leq n-5$ and hence $e(G) \leq \left\lceil \frac{(n-5)(n-1+r)}{2} \right\rceil$.

Now suppose that G is not connected and $G = G_1 \cup \dots \cup G_s$, where G_i is a component of G with $|V(G_i)| = p_i$ and $p_1 \leq p_2 \leq \dots \leq p_s$. By Lemma 4.1 and the argument in the proof of Lemma 2.8, we have $s = 2$ and $p_1 \leq r$. Set $r' = r - p_1$. Then $0 \leq r' = r - p_1 \leq n-9-p_1 \leq n-10$. For $r' \geq 2$ we have $r'(n-8-r') \geq 2(n-10) > 7 > 5 + ((-1)^n - (-1)^{(n-1)(r'-1)})/2$. For $r' = 1$ we have $r'(n-8-r') = n-9 > 5 + ((-1)^n - (-1)^{(n-1)(r'-1)})/2$. Since $|V(G_2)| = p_2 = n-1+r-p_1 = n-1+r'$, using Lemma 4.2 we see that for $r' \geq 1$ we have

$\Delta(G_2) \leq n - 5$ and so $e(G_2) \leq \lfloor \frac{(n-5)(n-1+r-p_1)}{2} \rfloor$. From Lemmas 4.1 and 2.5 we see that $p_1 \leq n - 7$. Hence, for $p_1 < r$,

$$\begin{aligned} e(G) &= e(G_1) + e(G_2) \leq \frac{p_1(p_1 - 1)}{2} + \lfloor \frac{(n-5)(n-1+r-p_1)}{2} \rfloor \\ &= \lfloor \frac{(n-5)(n-1+r) - p_1(n-4-p_1)}{2} \rfloor \leq \lfloor \frac{(n-5)(n-1+r) - 3p_1}{2} \rfloor \\ &< \lfloor \frac{(n-5)(n-1+r)}{2} \rfloor. \end{aligned}$$

This contradicts the fact that $e(G) = \text{ex}(n-1+r; T_n^3) \geq \lfloor \frac{(n-5)(n-1+r)}{2} \rfloor$. Thus, $p_1 = r$ and so $e(G) = e(K_{n-1} \cup K_r) = \binom{n-1}{2} + \binom{r}{2}$.

By the above, we always have

$$\text{ex}(n-1+r; T_n^3) = e(G) \leq \max \left\{ \left\lfloor \frac{(n-5)(n-1+r)}{2} \right\rfloor, \binom{n-1}{2} + \binom{r}{2} \right\}.$$

Thus the result is true.

Theorem 4.2. *Let $p, n \in \mathbb{N}$, $p \geq n \geq 15$, $p = k(n-1) + r$, $k \in \mathbb{N}$ and $r \in \{3, 4, \dots, n-9\}$. Then*

$$\text{ex}(p; T_n^3) = \frac{(n-2)p - r(n-1-r)}{2} + \max \left\{ 0, \left\lfloor \frac{r(n-4-r) - 3(n-1)}{2} \right\rfloor \right\}.$$

Proof. By Lemma 4.3,

$$\begin{aligned} &\text{ex}(n-1+r; T_n^3) \\ &= \max \left\{ \left\lfloor \frac{(n-5)(n-1+r)}{2} \right\rfloor, \binom{n-1}{2} + \binom{r}{2} \right\} \\ &= \frac{(n-2)(n-1+r) - r(n-1-r)}{2} + \max \left\{ 0, \left\lfloor \frac{r(n-4-r) - 3(n-1)}{2} \right\rfloor \right\}. \end{aligned}$$

Thus the result is true for $p = n-1+r < 2n-2$.

Now assume $p \geq 2n-2$. From the above and Lemmas 4.1 and 2.7 we see that

$$\begin{aligned} &\text{ex}(p; T_n^3) \\ &= \frac{(n-2)(p - (n-1+r))}{2} + \text{ex}(n-1+r; T_n^3) \\ &= \frac{(n-2)(p - (n-1+r))}{2} + \max \left\{ \left\lfloor \frac{(n-5)(n-1+r)}{2} \right\rfloor, \binom{n-1}{2} + \binom{r}{2} \right\} \\ &= \frac{(n-2)p - r(n-1-r)}{2} + \max \left\{ 0, \left\lfloor \frac{r(n-4-r) - 3(n-1)}{2} \right\rfloor \right\}. \end{aligned}$$

This completes the proof.

Lemma 4.4. *Let $m, n \in \mathbb{N}$ with $m \leq n-4$ and $n \geq 10$. Suppose that $G \in \text{Ex}(2n-6-m; T_n^3)$ and G is connected. Assume that $v_0 \in V(G)$ and $d(v_0) = \Delta(G) = n-4$. Then for any $v \in V(G) - \{v_0\} \cup \Gamma(v_0)$ we have $d(v) \leq n-5$.*

Proof. Assume that $\Gamma(v_0) = \{v_1, \dots, v_{n-4}\}$ and $\Gamma_2(v_0) = \{u_1, \dots, u_t\}$. Clearly $t \leq n-3-m \leq n-4-m \leq n-5$ for $v \in V(G) - \{v_0, v_1, \dots, v_{n-4}, u_1, \dots, u_t\}$. Thus, we only need to prove that $d(u_i) \leq n-5$ for $i =$

$1, 2, \dots, t$. We may suppose that v_1, \dots, v_{s_1} are all vertices adjacent to exactly two vertices in the set $\{u_1, \dots, u_t\}$ and $v_{s_1+1}, \dots, v_{s_2}$ are all vertices adjacent to exactly one vertex in the set $\{u_1, \dots, u_t\}$. Let $V_1 = \{v_0, v_1, \dots, v_{n-4}\}$ and $V_1' = V(G) - V_1$. Since $n - 5 - m \leq n - 6$, by (4.2) we have $e(G[V_1]) \leq \frac{n^2 - 7n + 12 - s_1 - 2s_2}{2}$. Suppose $d(u_i) = n - 4$ for some $i \in \{1, \dots, t\}$ and $\Gamma(u_i) \cap \{v_1, \dots, v_{n-4}\} = \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$. Then $m \leq k \leq n - 4$. If $k \geq \frac{n-8}{2}$, then $d_{G-V_1}(u_i) \leq n - 4 - \frac{n-8}{2} = \frac{n}{2}$ and so

$$\begin{aligned} e(G - V_1) &\leq d_{G-V_1}(u_i) + \binom{n-4-m}{2} \\ &\leq \frac{n}{2} + \frac{(n-4-m)(n-5-m)}{2} = \frac{n^2 - (2m+8)n + (m+4)(m+5)}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} e(G) &= e(G[V_1]) + e(V_1 V_1') + e(G - V_1) \\ &\leq \frac{n^2 - 7n + 12 - s_1 - 2s_2}{2} + 2s_1 + s_2 - s_1 + \frac{n^2 - (2m+8)n + (m+4)(m+5)}{2} \\ &= \frac{2n^2 - (2m+15)n + m^2 + 9m + 32 + s_1}{2} \\ &\leq \frac{2n^2 - (2m+15)n + m^2 + 9m + 32 + n - 4}{2} = n^2 - (m+7)n + \frac{m(m+9)}{2} + 14 \\ &< n^2 - (m+7)n + \frac{m(m+11)}{2} + 16 = e(K_{n-1} \cup K_{n-5-m}). \end{aligned}$$

This is a contradiction. Hence $k < \frac{n-8}{2}$. As $d(u_i) = n - 4$ and G does not contain T_n^3 as a subgraph, for $j \in \{i_1, \dots, i_k\}$ we see that $|\Gamma(v_j) \cap (\{v_1, \dots, v_{n-4}\} - \{v_{i_1}, \dots, v_{i_k}\})| \leq 1$. Hence $d_{G[V_1]}(v_j) \leq k - 1 + 1 + 1 = k + 1$. As $d(v_0) = n - 4$ and $v_0 v_j \in E(G)$ we have $|\Gamma(v_j) \cap \{u_1, \dots, u_t\}| \leq 2$. Thus, $d(v_j) \leq k + 3$ and so $d(v_{i_1}) + \dots + d(v_{i_k}) \leq k(k+3)$. For $m \leq k < \frac{n-8}{2}$ we see that

$$k(n-7-k) - m(n-7-m) = (k-m)(n-7-k-m) \geq (k-m)(n-7 - \frac{n-7}{2} - m) \geq 0.$$

Thus, $k(n-7-k) \geq m(n-7-m)$ and so

$$\begin{aligned} e(G) &= \frac{1}{2} \sum_{v \in V(G)} d(v) \leq \frac{(2n-6-m-k)(n-4) + k(k+3)}{2} \\ &= \frac{(2n-6-m)(n-4) - k(n-7-k)}{2} \\ &\leq \frac{(2n-6-m)(n-4) - m(n-7-m)}{2} \\ &= n^2 - (m+7)n + \frac{m(m+11)}{2} + 12 \\ &< n^2 - (m+7)n + \frac{m(m+11)}{2} + 16 = e(K_{n-1} \cup K_{n-5-m}). \end{aligned}$$

This contradicts the assumption $G \in Ex(2n-6-m; T_n^3)$. Hence $d(u_i) \leq n-5$ for $i = 1, 2, \dots, t$ as claimed. The proof is now complete.

Lemma 4.5. *Let $n \in \mathbb{N}$ with $n \geq 10$. Then $ex(2n-7; T_n^3) = n^2 - 8n + 22$.*

Proof. Let $G \in Ex(2n-7; T_n^3)$. As $K_{n-1} \cup K_{n-6}$ does not contain T_n^3 as a subgraph, we have $e(G) \geq e(K_{n-1} \cup K_{n-6})$. We first assume that G is connected. By Lemma 4.1, $\Delta(G) = n-5$ or $n-4$. If $\Delta(G) = n-5$, then

$$e(G) \leq \frac{(n-5)(2n-7)}{2} = n^2 - \frac{17}{2}n + \frac{35}{2} < n^2 - 8n + 22 = e(K_{n-1} \cup K_{n-6}),$$

which contradicts the fact $e(G) \geq e(K_{n-1} \cup K_{n-6})$. Hence $\Delta(G) = n-4$. Suppose that $v_0 \in V(G)$, $d(v_0) = n-4$ and $\Gamma(v_0) = \{v_1, \dots, v_{n-4}\}$. By Lemma 4.4, $d(v) \leq n-5$ for $v \in V(G) - \{v_0, v_1, \dots, v_{n-4}\}$. Therefore,

$$\begin{aligned} e(G) &= \frac{1}{2} \sum_{v \in V(G)} d(v) \leq \frac{(n-3)(n-4) + (n-4)(n-5)}{2} \\ &= n^2 - 8n + 16 < n^2 - 8n + 22 = e(K_{n-1} \cup K_{n-6}). \end{aligned}$$

This is also a contradiction. So G is not connected.

Suppose that G is not connected and $G = G_1 \cup \dots \cup G_s$, where G_i is a component of G with $|V(G_i)| = p_i$ and $p_1 \leq p_2 \leq \dots \leq p_s$. By Lemmas 4.1, 2.5 and the argument in the proof of Lemma 2.8, we have $s = 2$ and $p_1 \leq n-6$. If $p_1 \leq n-7$, then $p_2 = 2n-7-p_1 \geq n$. By Lemmas 4.1 and 2.5, we have $2n-7 = p_1 + p_2 \leq 2n-8$. This is impossible. Hence $p_1 = n-6$, $p_2 = n-1$ and so $e(G) = e(K_{n-1} \cup K_{n-6}) = n^2 - 8n + 22$. This proves the lemma.

Theorem 4.3. *Let $p, n \in \mathbb{N}$, $p \geq n \geq 10$ and $p = k(n-1) + n-6$ with $k \in \mathbb{N}$. Then*

$$ex(p; T_n^3) = \frac{(n-2)p - 5(n-6)}{2}.$$

Proof. By Lemmas 4.1, 2.7 and 4.5,

$$\begin{aligned} ex(p; T_n^3) &= \frac{(n-2)(p - (2n-7))}{2} + ex(2n-7; T_n) \\ &= \frac{(n-2)(p - (2n-7))}{2} + n^2 - 8n + 22 = \frac{(n-2)p - 5(n-6)}{2}. \end{aligned}$$

Lemma 4.6. *Let $n \in \mathbb{N}$ with $n \geq 15$. Then*

$$ex(2n-9; T_n^3) = n^2 - 10n + 24 + \max \left\{ \left\lceil \frac{n}{2} \right\rceil, 13 \right\}.$$

Proof. Let $G \in Ex(2n-9; T_n^3)$. Suppose that G is not connected and $G = G_1 \cup \dots \cup G_s$, where G_i is a component of G with $|V(G_i)| = p_i$ and $p_1 \leq p_2 \leq \dots \leq p_s$. By Lemmas 4.1, 2.5 and the argument in the proof of Lemma 2.8, we have $s = 2$ and $p_1 \leq n-8$. If $p_1 \leq n-9$, then $p_2 = 2n-9-p_1 \geq n$. By Lemmas 4.1 and 2.5, we have $p_1(n-3-p_1) \leq p_1 + p_2 + 1 = 2n-8$. For $3 \leq p_1 \leq n-9$ we have $p_1(n-3-p_1) \geq 3(n-6) > 2n-8$. Thus, $p_1 = 1$ or 2 . For $p_1 = 1$ we have $p_2 = 2n-10 = n-1 + n-9$. As $(n-9)(n-8-(n-9)) = n-9 > 5 + ((-1)^n - (-1)^{(n-1)(n-10)})/2$, using Lemma 4.2 we see that $\Delta(G_2) \leq n-5$ and hence $e(G) = e(G_2) = ex(2n-10; K_{1,n-4}) = \frac{(2n-10)(n-5)}{2} = n^2 - 10n + 25$. For $p_1 = 2$ we have $p_2 = 2n-11 = n-1 + n-10$. As $(n-10)(n-8-(n-10)) = 2(n-10) > 7 > 5 + ((-1)^n - (-1)^{(n-1)(n-11)})/2$, using Lemma 4.2 we see that $\Delta(G_2) \leq n-5$

and hence $e(G_2) = \text{ex}(2n - 11; K_{1,n-4}) = \lfloor \frac{(2n-11)(n-5)}{2} \rfloor = \lfloor \frac{2n^2-21n+55}{2} \rfloor$. Thus, $e(G) = e(G_1) + 2(G_2) = 1 + \lfloor \frac{2n^2-21n+55}{2} \rfloor = \lfloor \frac{2n^2-21n+57}{2} \rfloor$. For $p_1 = n - 8$ we have $p_2 = n - 1$ and so $e(G) = e(K_{n-1} \cup K_{n-8}) = \binom{n-1}{2} + \binom{n-8}{2} = n^2 - 10n + 37$. Therefore, when G is not connected, we have

$$e(G) = \max \left\{ n^2 - 10n + 25, \left\lfloor \frac{2n^2 - 21n + 57}{2} \right\rfloor, n^2 - 10n + 37 \right\} = n^2 - 10n + 37.$$

Assume that G is connected. By Lemma 4.1, $\Delta(G) \leq n - 4$. If $\Delta(G) \leq n - 5$, then clearly $e(G) = \text{ex}(2n - 9; K_{1,n-4}) = \lfloor \frac{(2n-9)(n-5)}{2} \rfloor = \lfloor \frac{2n^2-19n+45}{2} \rfloor$. Now assume $\Delta(G) = n - 4$. Suppose $v_0 \in V(G)$, $d(v_0) = n - 4$, $\Gamma(v_0) = \{v_1, \dots, v_{n-4}\}$ and $\Gamma_2(v_0) = \{u_1, \dots, u_t\}$. Then clearly $t \leq n - 6$. We may suppose that v_1, \dots, v_{s_1} are all vertices adjacent to exactly two vertices in $\Gamma_2(v_0)$ and $v_{s_1+1}, \dots, v_{s_2}$ are all vertices adjacent to exactly one vertex in $\Gamma_2(v_0)$. Let $V_1 = \{v_0, v_1, \dots, v_{n-4}\}$ and $V'_1 = V(G) - V_1$. By Lemma 4.4, we have $d(v) \leq n - 5$ for $v \in V(G) - \{v_0, v_1, \dots, v_{n-4}\}$.

If $s_2 \geq n - 6$, from (4.1) we see that $d(v_i) \leq n - 5$ for $i = 1, 2, \dots, n - 6$. Since $d(v) \leq n - 5$ for all $v \in V(G) - \{v_0, \dots, v_{n-4}\}$ we see that

$$\begin{aligned} 2e(G) &= \sum_{v \in V(G)} d(v) \leq d(v_0) + d(v_{n-4}) + d(v_{n-5}) + (2n - 12)(n - 5) \\ &\leq 3(n - 4) + (2n - 12)(n - 5) = 2n^2 - 19n + 48 \end{aligned}$$

Thus, $e(G) \leq \lfloor n^2 - \frac{19}{2}n + 24 \rfloor$. If $s_2 < n - 6$, then $s_1 \leq s_2 < n - 6$. Using (4.2) we see that

$$\begin{aligned} e(G) &= e(G[V_1]) + e(V_1 V'_1) + e(G - V_1) \\ &\leq \frac{n^2 - 7n + 12 - s_1 - 2s_2}{2} + 2s_1 + s_2 - s_1 + \binom{n-6}{2} \\ &= \frac{2n^2 - 20n + 54 + s_1}{2} < \frac{2n^2 - 20n + 54 + n - 6}{2} = n^2 - \frac{19}{2}n + 24. \end{aligned}$$

Thus, we always have $e(G) \leq \lfloor n^2 - \frac{19}{2}n + 24 \rfloor = n^2 - 10n + 24 + \lfloor \frac{n}{2} \rfloor$.

When $n < 26$ we have $n^2 - \frac{19}{2}n + 24 < n^2 - 10n + 37 = e(K_{n-1} \cup K_{n-8})$. By the above, $\text{ex}(2n - 9; n) = n^2 - 10n + 37$.

Now we assume $n \geq 26$. Clearly $e(K_{n-1} \cup K_{n-8}) = n^2 - 10n + 37 \leq n^2 - \frac{19}{2}n + 24$. To prove the result, now we only need to construct a connected graph G_0 of order $2n - 9$ such that G_0 does not contain T_n^3 as a subgraph and $e(G_0) = n^2 - 10n + 24 + \lfloor \frac{n}{2} \rfloor$. When n is even, we may construct a regular graph H with degree $n - 10$ and $V(H) = \{v_1, \dots, v_{n-6}\}$. Let G_0 be a graph given by $V(G_0) = \{v_0, v_1, \dots, v_{n-4}, u_1, \dots, u_{n-6}\}$ and

$$\begin{aligned} E(G_0) &= E(H) \cup \{v_0 v_1, \dots, v_0 v_{n-4}, v_1 v_{n-5}, \dots, v_{n-6} v_{n-5}, v_1 v_{n-4}, \dots, v_{n-5} v_{n-4}, \\ &\quad v_1 u_1, v_1 u_2, v_2 u_1, v_2 u_2, \dots, v_{n-7} u_{n-7}, v_{n-7} u_{n-6}, v_{n-6} u_{n-7}, v_{n-6} u_{n-6}, \\ &\quad u_1 u_2, \dots, u_1 u_{n-6}, u_2 u_3, \dots, u_2 u_{n-6}, u_3 u_{n-6}, \dots, u_{n-7} u_{n-6}\}. \end{aligned}$$

Then $d(v_0) = d(v_{n-5}) = d(v_{n-4}) = n - 4$ and $d(v_1) = \dots = d(v_{n-6}) = d(u_1) = \dots = d(u_{n-6}) = n - 5$. Clearly G_0 does not contain any copies of T_n^3 and

$$e(G_0) = \frac{1}{2} \sum_{v \in V(G_0)} d(v) = \frac{3(n - 4) + (2n - 12)(n - 5)}{2} = n^2 - \frac{19}{2}n + 24.$$

When n is odd, let H be a graph with $V(H) = \{v_1, \dots, v_{n-6}\}$ and

$$E(H) = \{v_1v_2, v_2v_3, \dots, v_{n-7}v_{n-6}, v_{n-6}v_1, v_1v_{\frac{n-5}{2}}, v_2v_{\frac{n-3}{2}}, \dots, v_{\frac{n-7}{2}}v_{n-7}\}.$$

Then $d_H(v_1) = \dots = d_H(v_{n-7}) = 3$ and $d_H(v_{n-6}) = 2$. Let G_0 be a graph with $V(G_0) = \{v_0, v_1, \dots, v_{n-4}, u_1, \dots, u_{n-6}\}$ and

$$\begin{aligned} E(G_0) &= E(\overline{H}) \cup \{v_0v_1, \dots, v_0v_{n-4}, v_1v_{n-5}, \dots, v_{n-6}v_{n-5}, v_1v_{n-4}, \dots, v_{n-5}v_{n-4}, \\ &\quad v_1u_1, v_1u_2, v_2u_1, v_2u_2, \dots, v_{n-8}u_{n-8}, v_{n-8}u_{n-7}, v_{n-7}u_{n-8}, v_{n-7}u_{n-7}, v_{n-6}u_{n-6}, \\ &\quad u_1u_2, \dots, u_1u_{n-6}, u_2u_3, \dots, u_2u_{n-6}, u_3u_{n-6}, \dots, u_{n-7}u_{n-6}\}. \end{aligned}$$

Then $d(v_0) = d(v_{n-5}) = d(v_{n-4}) = n-4$, $d(v_1) = \dots = d(v_{n-6}) = d(u_1) = \dots = d(u_{n-7}) = n-5$ and $d(u_{n-6}) = n-6$. Clearly G_0 does not contain any copies of T_n^3 and

$$\begin{aligned} e(G_0) &= \frac{1}{2} \sum_{v \in V(G_0)} d(v) = \frac{3(n-4) + (n-6+n-7)(n-5) + n-6}{2} \\ &= \frac{2n^2 - 19n + 47}{2} = n^2 - 10n + 24 + \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned}$$

By the above, the lemma is proved.

Theorem 4.4. *Let $p, n \in \mathbb{N}$, $p \geq n \geq 15$ and $p = k(n-1) + n-8$ with $k \in \mathbb{N}$. Then*

$$ex(p; T_n^3) = \frac{(n-2)p - 7n + 30}{2} + \max \left\{ \left\lfloor \frac{n}{2} \right\rfloor, 13 \right\}.$$

Proof. By Lemmas 4.1, 2.7 and 4.6,

$$\begin{aligned} ex(p; T_n^3) &= \frac{(n-2)(p - (2n-9))}{2} + ex(2n-9; T_n) \\ &= \frac{(n-2)(p - (2n-9))}{2} + n^2 - 10n + 24 + \max \left\{ \left\lfloor \frac{n}{2} \right\rfloor, 13 \right\} \\ &= \frac{(n-2)p - 7n + 30}{2} + \max \left\{ \left\lfloor \frac{n}{2} \right\rfloor, 13 \right\}. \end{aligned}$$

Lemma 4.7. *Let $n \in \mathbb{N}$ with $n \geq 15$. Then*

$$ex(2n-8; T_n^3) = n^2 - 9n + 29 + \max \left\{ 0, \left\lfloor \frac{n-37}{4} \right\rfloor \right\}.$$

Proof. Let $G \in Ex(2n-8; T_n^3)$. Then clearly $e(G) \geq e(K_{n-1} \cup K_{n-7}) = n^2 - 9n + 29$. Suppose that G is not connected and $G = G_1 \cup \dots \cup G_s$, where G_i is a component of G with $|V(G_i)| = p_i$ and $p_1 \leq p_2 \leq \dots \leq p_s$. By Lemmas 4.1, 2.5 and the argument in the proof of Lemma 2.8, we have $s = 2$ and $p_1 \leq n-7$. If $p_1 = n-7$, then $p_2 = n-1$ and so $e(G) = e(K_{n-7} \cup K_{n-1}) = n^2 - 9n + 29$. If $p_1 \leq n-8$, then $p_2 = 2n-8-p_1 \geq n$. By Lemmas 4.1 and 2.5, we have $p_1(n-3-p_1) \leq 2n-7$. For $3 \leq p_1 \leq n-8$ we have $p_1(n-3-p_1) \geq 3(n-6) > 2n-7$. Thus, $p_1 = 1$ or 2 . For $p_1 = 2$ we have $p_2 = 2n-10 = n-1+n-9$. As $(n-9)(n-8-(n-9)) = n-9 \geq 6 > 5 + ((-1)^n - (-1)^{(n-1)(n-10)})/2$, using Lemma 4.2 we see that $\Delta(G_2) \leq n-5$

and hence $e(G_2) = \text{ex}(2n-10; K_{1,n-4}) = \lfloor \frac{(2n-10)(n-5)}{2} \rfloor = n^2 - 10n + 25$. Thus, $e(G) = e(G_1) + 2e(G_2) = 1 + n^2 - 10n + 25 = n^2 - 10n + 26 < n^2 - 9n + 29 = e(K_{n-7} \cup K_{n-1})$. This is impossible. For $p_1 = 1$ we have $p_2 = 2n - 9$, from the proof of Lemma 4.6 we see that $n \geq 26$ and $e(G) = e(G_2) = n^2 - 10n + 24 + \lfloor \frac{n}{2} \rfloor < n^2 - 9n + 29 = e(K_{n-7} \cup K_{n-1})$. This is also impossible. Therefore, when G is not connected, we have $e(G) = e(K_{n-7} \cup K_{n-1}) = n^2 - 9n + 29$.

Assume that G is connected. By Lemma 4.1, $\Delta(G) \leq n - 4$. If $\Delta(G) \leq n - 5$, then clearly $e(G) = \text{ex}(2n-8; K_{1,n-4}) = \lfloor \frac{(2n-8)(n-5)}{2} \rfloor = n^2 - 9n + 20 < n^2 - 9n + 29 = e(K_{n-7} \cup K_{n-1})$. This is a contradiction. Hence $\Delta(G) = n - 4$. Suppose $v_0 \in V(G)$, $d(v_0) = n - 4$, $\Gamma(v_0) = \{v_1, \dots, v_{n-4}\}$ and $\Gamma_2(v_0) = \{u_1, \dots, u_t\}$. Then clearly $t \leq n - 5$. We may suppose that v_1, \dots, v_{s_1} are all vertices adjacent to exactly two vertices in $\Gamma_2(v_0)$ and $v_{s_1+1}, \dots, v_{s_2}$ are all vertices adjacent to exactly one vertex in $\Gamma_2(v_0)$. Let $V_1 = \{v_0, v_1, \dots, v_{n-4}\}$ and $V'_1 = V(G) - V_1$. By (4.1), $d(v_i) \leq n - 5$ for $i = 1, 2, \dots, s_2$. By Lemma 4.4, $d(v) \leq n - 5$ for $v \in V'_1$. Thus,

$$\begin{aligned} e(G) &= \frac{1}{2} \sum_{v \in V(G)} d(v) \leq \frac{(n-3-s_2)(n-4) + (n-5+s_2)(n-5)}{2} \\ &= \frac{2n^2 - 17n + 37 - s_2}{2} = n^2 - 9n + 29 + \frac{n-21-s_2}{2}. \end{aligned}$$

Since $e(G) \geq n^2 - 9n + 29$ we get $s_2 \leq n - 21$. By (4.2), $e(G[V_1]) \leq \frac{n^2 - 7n + 12 - s_1 - 2s_2}{2}$. Thus,

$$\begin{aligned} e(G) &= e(G[V_1]) + e(V_1 V'_1) + e(G - V_1) \\ &\leq \frac{n^2 - 7n + 12 - s_1 - 2s_2}{2} + 2s_1 + s_2 - s_1 + \binom{n-5}{2} = n^2 - 9n + 21 + \frac{s_1}{2} \\ &\leq n^2 - 9n + 21 + \frac{n-21}{2} = n^2 - \frac{17}{2}n + \frac{21}{2}. \end{aligned}$$

As $e(G) \geq n^2 - 9n + 29$ we get $n^2 - \frac{17}{2}n + \frac{21}{2} \geq n^2 - 9n + 29$ and so $n \geq 37$.

If $s_1 \leq \frac{n-5}{2}$, from the above we see that

$$e(G) \leq n^2 - 9n + 21 + \frac{n-5}{4} = n^2 - 9n + 29 + \frac{n-37}{4}.$$

If $s_1 = \frac{n-5}{2} + s'_1 > \frac{n-5}{2}$, then $e(V_1 V'_1) \geq 2s_1 = n - 5 + 2s'_1$. As $d(v) \leq n - 5$ for $v \in V'_1$, we see that $e(V_1 V'_1) + 2e(G - V_1) = \sum_{v \in V'_1} d(v) \leq (n-5)(n-5)$ and so $2e(G - V_1) \leq (n-5)^2 - e(V_1 V'_1) \leq (n-5)^2 - (n-5) - 2s'_1 = n^2 - 11n + 30 - 2s'_1$. Therefore,

$$\begin{aligned} e(G) &= e(G[V_1]) + e(V_1 V'_1) + e(G - V_1) \\ &\leq \frac{n^2 - 7n + 12 - s_1 - 2s_2}{2} + 2s_1 + s_2 - s_1 + \frac{n^2 - 11n + 30 - 2s'_1}{2} \\ &= n^2 - 9n + 29 + \frac{n-37-2s'_1}{4} < n^2 - 9n + 29 + \frac{n-37}{4}. \end{aligned}$$

Thus, when G is connected, we always have $n \geq 37$ and $e(G) \leq n^2 - 9n + 29 + \lfloor \frac{n-37}{4} \rfloor$.

By the above, for $n < 37$ we see that G is not connected and $e(G) = n^2 - 9n + 29$. Now assume $n \geq 37$. Then $n^2 - 9n + 29 + \lfloor \frac{n-37}{4} \rfloor \geq n^2 - 9n + 29$. To prove the

result, we only need to construct a connected graph G_0 of order $n^2 - 9n + 29 + \lceil \frac{n-37}{4} \rceil$ without T_n^3 . Let us consider the following four cases:

Case 1. $n \equiv 1 \pmod{4}$. In this case, by [3, Corollary 2.1] we may construct a regular graph H with degree $\frac{n-13}{2}$ and $V(H) = \{v_1, \dots, v_{\frac{n-5}{2}}\}$. Let G_0 be a graph with $V(G_0) = \{v_0, v_1, \dots, v_{n-4}, u_1, \dots, u_{n-5}\}$ and

$$\begin{aligned} E(G_0) = E(H) \cup \{ & u_1 u_2, \dots, u_1 u_{n-5}, u_2 u_3, \dots, u_2 u_{n-5}, u_3 u_4, \dots, u_{n-6} u_{n-5}, \\ & v_1 u_1, v_1 u_2, \dots, v_{\frac{n-5}{2}} u_{n-6}, v_{\frac{n-5}{2}} u_{n-5}, v_0 v_1, \dots, v_0 v_{n-4}, \\ & v_1 v_{\frac{n-3}{2}}, \dots, v_1 v_{n-4}, \dots, v_{\frac{n-5}{2}} v_{\frac{n-3}{2}}, \dots, v_{\frac{n-5}{2}} v_{n-4}, \\ & v_{\frac{n-3}{2}} v_{\frac{n-1}{2}}, \dots, v_{\frac{n-3}{2}} v_{n-4}, v_{\frac{n-1}{2}} v_{\frac{n+1}{2}}, \dots, v_{n-5} v_{n-4} \}. \end{aligned}$$

Then $d(v_0) = d(v_{\frac{n-3}{2}}) = \dots = d(v_{n-4}) = n-4$ and $d(v_1) = \dots = d(v_{\frac{n-5}{2}}) = d(u_1) = \dots = d(u_{n-5}) = n-5$. It is clear that G_0 does not contain any copies of T_n^3 and

$$\begin{aligned} 2e(G_0) &= \sum_{v \in V(G_0)} d(v) = \left(n-5 + \frac{n-5}{2}\right)(n-5) + \left(n-3 - \frac{n-5}{2}\right)(n-4) \\ &= 2n^2 - 18n + 58 + \frac{n-37}{2}. \end{aligned}$$

Therefore, $e(G_0) = n^2 - 9n + 29 + \frac{n-37}{4}$.

Case 2. $n \equiv 2 \pmod{4}$. Let H be a graph with $V(H) = \{v_1, \dots, v_{\frac{n-4}{2}}\}$ and

$$E(H) = \{v_1 v_2, v_2 v_3, \dots, v_{\frac{n-6}{2}} v_{\frac{n-4}{2}}, v_{\frac{n-4}{2}} v_1, v_1 v_{\frac{n-2}{4}}, v_2 v_{\frac{n+2}{4}}, \dots, v_{\frac{n-6}{4}} v_{\frac{n-6}{2}}\}.$$

Then $d_H(v_1) = \dots = d_H(v_{\frac{n-6}{2}}) = 3$ and $d_H(v_{\frac{n-4}{2}}) = 2$. Let G_0 be a graph with $V(G_0) = \{v_0, v_1, \dots, v_{n-4}, u_1, \dots, u_{n-5}\}$ and

$$\begin{aligned} E(G_0) = E(\overline{H}) \cup \{ & u_i u_j \ (i, j \in \{1, 2, \dots, n-5\}, i < j), \\ & v_1 u_1, v_1 u_2, \dots, v_{\frac{n-6}{2}} u_{n-7}, v_{\frac{n-6}{2}} u_{n-6}, v_{\frac{n-4}{2}} u_{n-5}, \\ & v_0 v_1, \dots, v_0 v_{n-4}, v_i v_j \ (i \in \{(n-2)/2, \dots, n-4\}, j \in \{1, 2, \dots, i-1\}) \}. \end{aligned}$$

Then $d(v_0) = d(v_{\frac{n-2}{2}}) = \dots = d(v_{n-4}) = n-4$ and $d(v_1) = \dots = d(v_{\frac{n-4}{2}}) = d(u_1) = \dots = d(u_{n-5}) = n-5$. It is clear that G_0 does not contain any copies of T_n^3 and

$$\begin{aligned} 2e(G_0) &= \left(n-5 + \frac{n-4}{2}\right)(n-5) + \left(n-3 - \frac{n-4}{2}\right)(n-4) \\ &= 2n^2 - 18n + 58 + \frac{n-38}{2}. \end{aligned}$$

Therefore, $e(G_0) = n^2 - 9n + 29 + \lceil \frac{n-37}{4} \rceil$.

Case 3. $n \equiv 3 \pmod{4}$. Let H be a graph with $V(H) = \{v_1, \dots, v_{\frac{n-3}{2}}\}$ and

$$E(H) = \{v_1 v_2, v_2 v_3, \dots, v_{\frac{n-5}{2}} v_{\frac{n-3}{2}}, v_{\frac{n-3}{2}} v_1, v_1 v_{\frac{n-3}{4}}, v_2 v_{\frac{n+1}{4}}, \dots, v_{\frac{n-7}{4}} v_{\frac{n-7}{2}}\}.$$

Then $d_H(v_1) = \dots = d_H(v_{\frac{n-7}{2}}) = 3$ and $d_H(v_{\frac{n-5}{2}}) = d_H(v_{\frac{n-3}{2}}) = 2$. Let G_0 be a graph with $V(G_0) = \{v_0, v_1, \dots, v_{n-4}, u_1, \dots, u_{n-5}\}$ and

$$E(G_0) = E(\overline{H}) \cup \{u_i u_j \ (i, j \in \{1, 2, \dots, n-5\}, i < j),$$

$$v_1u_1, v_1u_2, \dots, v_{\frac{n-7}{2}}u_{n-8}, v_{\frac{n-7}{2}}u_{n-7}, v_{\frac{n-5}{2}}u_{n-6}, v_{\frac{n-3}{2}}u_{n-5}, \\ v_0v_1, \dots, v_0v_{n-4}, v_iv_j (i \in \{(n-1)/2, \dots, n-4\}, j \in \{1, 2, \dots, i-1\})\}.$$

Then $d(v_0) = d(v_{\frac{n-1}{2}}) = \dots = d(v_{n-4}) = n-4$ and $d(v_1) = \dots = d(v_{\frac{n-3}{2}}) = d(u_1) = \dots = d(u_{n-5}) = n-5$. Clearly G_0 does not contain any copies of T_n^3 and

$$2e(G_0) = \left(n-5 + \frac{n-3}{2}\right)(n-5) + \left(n-3 - \frac{n-3}{2}\right)(n-4) \\ = 2n^2 - 18n + 58 + \frac{n-39}{2}.$$

Therefore, $e(G_0) = n^2 - 9n + 29 + \frac{n-39}{4} = n^2 - 9n + 29 + \lfloor \frac{n-37}{4} \rfloor$.

Case 4. $n \equiv 0 \pmod{4}$. Let H be a graph with $V(H) = \{v_1, \dots, v_{\frac{n-2}{2}}\}$ and

$$E(H) = \{v_1v_2, v_2v_3, \dots, v_{\frac{n-4}{2}}v_{\frac{n-2}{2}}, v_{\frac{n-2}{2}}v_1, v_1v_{\frac{n-4}{4}}, v_2v_{\frac{n}{4}}, \dots, v_{\frac{n-8}{4}}v_{\frac{n-2}{2}}\}.$$

Then $d_H(v_1) = \dots = d_H(v_{\frac{n-8}{2}}) = 3$ and $d_H(v_{\frac{n-6}{2}}) = d_H(v_{\frac{n-4}{2}}) = d_H(v_{\frac{n-2}{2}}) = 2$. Let G_0 be a graph with $V(G_0) = \{v_0, v_1, \dots, v_{n-4}, u_1, \dots, u_{n-5}\}$ and

$$E(G_0) = E(\overline{H}) \cup \{u_iu_j \mid (i, j \in \{1, 2, \dots, n-5\}, i < j), \\ v_1u_1, v_1u_2, \dots, v_{\frac{n-8}{2}}u_{n-9}, v_{\frac{n-8}{2}}u_{n-8}, v_{\frac{n-6}{2}}u_{n-7}, v_{\frac{n-4}{2}}u_{n-6}, v_{\frac{n-2}{2}}u_{n-5}, \\ v_0v_1, \dots, v_0v_{n-4}, v_iv_j (i \in \{n/2, \dots, n-4\}, j \in \{1, 2, \dots, i-1\})\}.$$

Then $d(v_0) = d(v_{\frac{n}{2}}) = \dots = d(v_{n-4}) = n-4$ and $d(v_1) = \dots = d(v_{\frac{n-2}{2}}) = d(u_1) = \dots = d(u_{n-5}) = n-5$. Clearly G_0 does not contain any copies of T_n^3 and

$$2e(G_0) = \left(n-5 + \frac{n-2}{2}\right)(n-5) + \left(n-3 - \frac{n-2}{2}\right)(n-4) \\ = 2n^2 - 18n + 58 + \frac{n-40}{2}.$$

Therefore, $e(G_0) = n^2 - 9n + 29 + \frac{n-40}{4} = n^2 - 9n + 29 + \lfloor \frac{n-37}{4} \rfloor$.

Summarizing the above we prove the lemma.

Theorem 4.5. Let $p, n \in \mathbb{N}$, $p \geq n \geq 15$ and $p = k(n-1) + n-7$ with $k \in \mathbb{N}$. Then

$$ex(p; T_n^3) = \frac{(n-2)p - 6(n-7)}{2} + \max \left\{ \left\lfloor \frac{n-37}{4} \right\rfloor, 0 \right\}.$$

Proof. By Lemmas 4.1, 2.7 and 4.7,

$$ex(p; T_n^3) = \frac{(n-2)(p - (2n-8))}{2} + ex(2n-8; T_n) \\ = \frac{(n-2)(p - (2n-8))}{2} + n^2 - 9n + 29 + \max \left\{ \left\lfloor \frac{n-37}{4} \right\rfloor, 0 \right\} \\ = \frac{(n-2)p - 6(n-7)}{2} + \max \left\{ \left\lfloor \frac{n-37}{4} \right\rfloor, 0 \right\}.$$

5. Evaluation of $ex(p; T_n''')$

Lemma 5.1. *Let $p, n \in \mathbb{N}$, $p \geq n \geq 10$ and $G \in Ex(p; T_n''')$. Suppose that G is connected. Then $\Delta(G) \leq n - 4$.*

Proof. By Lemma 2.1, $\Delta(G) \geq n - 5$. Suppose $v_0 \in V(G)$, $d(v_0) = \Delta(G) = m$ and $\Gamma(v_0) = \{v_1, \dots, v_m\}$. If $m = p - 1$, as G does not contain T_n''' as a subgraph, we see that $G[v_1, \dots, v_m]$ does not contain $3K_2$ as a subgraph. If $G[v_1, \dots, v_m]$ does not contain $2K_2$ as a subgraph, then G does not contain T_n''' as a subgraph, $e(G[v_1, \dots, v_m]) \leq e(K_{1, m-1}) = m - 1$ and so $e(G) = d(v_0) + e(G[v_1, \dots, v_m]) \leq m + m - 1 = 2m - 1$. Suppose that $G[v_1, \dots, v_m]$ contains $2K_2$ as a subgraph and $v_1v_2, v_3v_4 \in E(G)$. Then every edge in $G[v_1, \dots, v_m]$ is incident with some vertex in $\{v_1, v_2, v_3, v_4\}$. If $v_2v_i, v_3v_i, v_4v_i \in E(G)$ for some $i \in \{5, \dots, m\}$, then all edges in $E(G[v_1, \dots, v_m]) - \{v_3v_4\}$ is incident with v_2 or v_i and so $e(G) = d(v_0) + e(G[v_1, \dots, v_m]) \leq m + d(v_2) - 1 + d(v_i) - 1 \leq 3m - 2$. If $d(v_i) \leq 3$ for $i = 5, \dots, m$, then there are at most two vertices in $\{v_1, v_2, v_3, v_4\}$ adjacent to a fixed vertex in $\{v_5, \dots, v_m\}$ and so $e(G) \leq d(v_0) + 2(m - 4) + 2 = 3m - 6$. From the above we always have $e(G) \leq 3m$. This contradicts Lemma 2.2. Hence $m < p - 1$. Suppose that u_1, \dots, u_t are all vertices in G such that $d(u_1, v_0) = \dots = d(u_t, v_0) = 2$. Then $t \geq 1$. We may assume $u_1v_1 \in E(G)$ with no loss of generality. Set $V_1 = \{v_0, v_1, \dots, v_m\}$ and $V_2 = \{v_0, v_1, \dots, v_m, u_1\}$.

Suppose $t = 1$ and $m \geq n - 2$. Let $G' = G[v_2, v_3, \dots, v_m]$. As G does not contain T_n''' , we see that G' does not contain any copies of $2K_2$. If $e(G') \leq 2$, then

$$e(G) - e(G - V_1) \leq d(v_0) + d(u_1) + d(v_1) - 2 + e(G') \leq m + m + m - 2 + 2 = 3m,$$

which contradicts Lemma 2.2. Hence $e(G') \geq 3$. We claim that G' does not contain any copies of K_3 . We may assume $v_2v_3, v_2v_4, v_3v_4 \in E(G')$. As G' does not contain any copies of $2K_2$ we see that $e(G') = 3$. If $|\Gamma(u_1) \cap \{v_2, \dots, v_m\}| \geq 2$ and $|\Gamma(u_1) \cap \{v_5, \dots, v_m\}| \geq 1$, then v_1 is not adjacent to any vertex in G' . Hence $e(G) - e(G - V_1) \leq d(v_0) + d(u_1) + e(G') \leq m + m + 3 < 3m$. By Lemma 2.2, this is impossible. Similarly, if $|\Gamma(v_1) \cap \{v_2, \dots, v_m\}| \geq 2$ and $|\Gamma(v_1) \cap \{v_5, \dots, v_m\}| \geq 1$, then u_1 is not adjacent to any vertex in G' . Hence $e(G) - e(G - V_1) \leq d(v_0) + d(v_1) - 1 + e(G') \leq m + m - 1 + 3 < 3m$, which contradicts Lemma 2.2. As $m \geq n - 2 \geq 8$, $d(v_1) \leq m$ and $d(u_1) \leq m$, from the above we have

$$|\Gamma(v_1) \cap V(G')| + |\Gamma(u_1) \cap V(G')| \leq \max\{3 + 3, m - 1\} = m - 1.$$

Thus,

$$e(G) - e(G - V_1) \leq d(v_0) + 1 + |\Gamma(v_1) \cap V(G')| + |\Gamma(u_1) \cap V(G')| \leq m + 1 + m - 1 = 2m.$$

This is also impossible by Lemma 2.2. Hence the claim is true.

Now assume that $e(G') \geq 3$ and G' does not contain any copies of $2K_2$ and K_3 . Then all edges in G' have a common endpoint. We may assume that v_2 is such a vertex. Therefore $d_{G'}(v_2) \geq 3$. Suppose $v_1v_i \in E(G)$ for some $i \in \{3, 4, \dots, m\}$. Then $u_1v_j \notin E(G)$ for all $j \in \{3, 4, \dots, m\} - \{i\}$ and so $|\Gamma(u_1) \cap \{v_2, \dots, v_m\}| \leq 2$. Otherwise, for some $v_k \in \Gamma(v_2)$ the three edges v_1v_i, u_1v_j, v_2v_k induce a copy of $3K_2$ and so G contains a copy of T_n''' . Hence $d(v_1) + |\Gamma(u_1) \cap \{v_2, \dots, v_m\}| \leq m + 2$.

If $v_1v_i \notin E(G)$ for every $i \in \{3, 4, \dots, m\}$, then $d(v_1) \leq 3$ and $d(v_1) + |\Gamma(u_1) \cap \{v_2, \dots, v_m\}| \leq 3 + (m-1) = m+2$. Therefore,

$$\begin{aligned} e(G) - e(G - V_1) &= d(v_0) - 1 + d(v_1) + |\Gamma(u_1) \cap \{v_2, \dots, v_m\}| + d_{G'}(v_2) \\ &\leq m - 1 + (m + 2) + (m - 2) < 3m. \end{aligned}$$

This is impossible by Lemma 2.2. Hence $\Delta(G) \leq n - 3$ for $t = 1$.

Suppose $t = 1$ and $\Delta(G) = m \in \{n - 3, n - 4\}$. Then

$$\begin{aligned} e(G) - e(G - V_2) &= d(u_1) + e(G[v_0, v_1, \dots, v_m]) \\ &\leq m + e(K_{m+1}) = \frac{m^2 + 3m}{2} < \frac{(m+1)(m+2)}{2} = e(K_{m+2}). \end{aligned}$$

Thus, $e(G) < e((G - V_2) \cup K_{m+2})$, which contradicts the assumption $G \in Ex(p; T_n''')$.

By the above, for $t = 1$ we have $\Delta(G) \leq n - 5$. From now on we assume that $t \geq 2$. Suppose $t = 2$, $u_1v_1, u_2v_2 \in E(G)$ and $m = \Delta(G) \geq n - 3$. As G does not contain any copies of T_n''' , we see that $\{v_3, \dots, v_m\}$ is an independent set in G' . If $i, j \in \{3, 4, \dots, m\}$, $i \neq j$ and $v_1v_i, u_1v_j \in E(G)$, then u_2v_2, v_1v_i, u_1v_j induce a copy of $3K_2$ and so G contains a copy of T_n''' . Hence $|\Gamma(v_1) \cap \{v_3, \dots, v_m\}| + |\Gamma(u_1) \cap \{v_3, \dots, v_m\}| \leq m - 2$. Similarly, $|\Gamma(v_2) \cap \{v_3, \dots, v_m\}| + |\Gamma(u_2) \cap \{v_3, \dots, v_m\}| \leq m - 2$. If $u_1v_r, u_2v_s \in E(G)$, where $r, s \in \{3, 4, \dots, m\}$ and $r \neq s$, then $v_1v_2 \notin E(G)$, otherwise u_1v_r, u_2v_s, v_1v_2 induce a copy of $3K_2$ and so G contains a copy of T_n''' . Hence

$$\begin{aligned} e(G) - e(G - V_1) &= d(v_0) + e(G[v_1, v_2, u_1, u_2]) - e(G([u_1, u_2])) \\ &\quad + |\Gamma(v_1) \cap \{v_3, \dots, v_m\}| + |\Gamma(u_1) \cap \{v_3, \dots, v_m\}| \\ &\quad + |\Gamma(v_2) \cap \{v_3, \dots, v_m\}| + |\Gamma(u_2) \cap \{v_3, \dots, v_m\}| \\ &\leq m + 4 + (m - 2) + (m - 2) = 3m. \end{aligned}$$

By Lemma 2.2, $e(G) - e(G - V_1) > 3m$. We get a contradiction. Hence $\Delta(G) = m \leq n - 4$.

Suppose $t = 2$ and $v_1u_1, v_1u_2 \in E(G)$. If $u_iv_j \in E(G)$ for some $i \in \{1, 2\}$ and $j \in \{2, 3, \dots, m\}$, by the above argument we have $\Delta(G) \leq n - 4$. Now suppose that $u_iv_j \notin E(G)$ for every $i = 1, 2$ and $j = 2, 3, \dots, m$. If $m \geq n - 2$, then G' does not contain $2K_2$ as a subgraph. If G' contains a copy of K_3 , then $e(G') = 3$ and so

$$e(G) - e(G - V_1) = d(v_0) + d(v_1) - 1 + e(G') \leq m + m - 1 + 3 = 2m + 2.$$

This is impossible by Lemma 2.2. Thus all edges in G' have a common endpoint and so $e(G') \leq e(K_{1, m-2}) = m - 2$. Hence $e(G) - e(G - V_1) = d(v_0) + d(v_1) - 1 + e(G') \leq m + m - 1 + m - 2 = 3m - 3$. By Lemma 2.2, this is impossible. Therefore $m = \Delta(G) \leq n - 3$. If $m = n - 3$, then

$$\begin{aligned} e(G) - e(G - V_2) &= d(v_1) + d(u_1) - 1 + e(G[v_0, v_2, v_3, \dots, v_{n-3}]) \\ &\leq n - 3 + n - 3 - 1 + \binom{n-3}{2} = \frac{n^2 - 3n - 2}{2} \\ &< \frac{(n-1)(n-2)}{2} = e(K_{n-1}). \end{aligned}$$

Thus, $e(G) < e((G - V_2) \cup K_{n-1})$, which contradicts the assumption $G \in Ex(p; T_n''')$. Therefore $\Delta(G) \leq n - 4$.

From now on we assume $t \geq 3$. Suppose $|\Gamma(v_1) \cap \Gamma_2(v_0)| \geq 2$ and $|\Gamma(v_2) \cap \Gamma_2(v_0)| \geq 1$. If $|\Gamma(v_2) \cap \Gamma_2(v_0)| = 1$ and $v_2 u_2 \in E(G)$, then $\{v_3, \dots, v_m\}$ is an independent set in G' and $u_i v_j \notin E(G)$ for any $i \in \{1, 3, \dots, t\}$ and $j \in \{3, 4, \dots, m\}$. Suppose $v_2 v_i \in E(G)$ for some $i \in \{3, 4, \dots, m\}$. Then $u_2 v_j \notin E(G)$ for all $j \in \{3, 4, \dots, m\} - \{i\}$ and so $|\Gamma(u_2) \cap \{v_2, \dots, v_m\}| \leq 2$. Otherwise, the three edges $v_2 v_i, u_2 v_j, u_1 v_1$ induce a copy of $3K_2$ and so G contains a copy of T_n''' . Hence $d(v_2) + |\Gamma(u_2) \cap \{v_2, \dots, v_m\}| \leq m + 2$. If $v_2 v_i \notin E(G)$ for every $i \in \{3, 4, \dots, m\}$, then $d(v_2) \leq 3$ and $d(v_2) + |\Gamma(u_2) \cap \{v_2, \dots, v_m\}| \leq 3 + (m - 1) = m + 2$. Hence

$$e(G) - e(G - V_1) = d(v_0) + d(v_1) + d(v_2) - 2 + |\Gamma(u_2) \cap \{v_2, \dots, v_m\}| \leq 3m.$$

If $|\Gamma(v_1) \cap \Gamma_2(v_0)| \geq 2$ and $|\Gamma(v_2) \cap \Gamma_2(v_0)| \geq 2$, then $\{v_3, \dots, v_m\}$ is an independent set in G' and $u_i v_j \notin E(G)$ for any $i \in \{1, 2, \dots, t\}$ and $j \in \{3, 4, \dots, m\}$. Hence

$$e(G) - e(G - V_1) = d(v_0) + d(v_1) + d(v_2) - 2 \leq 3m - 2.$$

From the above we always have $e(G) - e(G - V_1) \leq 3m$, which contradicts Lemma 2.2. Therefore $m = \Delta(G) \leq n - 4$.

Lemma 5.2. *Let $p, n \in \mathbb{N}$, $p \geq n \geq 10$ and $G \in Ex(p; T_n''')$. Suppose that G is connected. Then $\Delta(G) = n - 5$.*

Proof. By Lemma 5.1, $\Delta(G) \leq n - 4$. Suppose that $v_0 \in V(G)$, $d(v_0) = \Delta(G) = n - 4$, $\Gamma(v_0) = \{v_1, \dots, v_{n-4}\}$ and $\Gamma_2(v_0) = \{u_1, \dots, u_t\}$. By the proof of Lemma 5.1, we have $\Delta(G) = n - 5$ for $t = 1$. Set $V_1 = \{v_0, v_1, \dots, v_{n-4}, u_1, u_2\}$.

Suppose $t = 2$. Then

$$\begin{aligned} e(G) - e(G - V_1) &= d(u_1) + d(u_2) + e(G[v_0, v_1, \dots, v_{n-4}]) \\ &\leq n - 4 + n - 4 + \frac{(n - 3)(n - 4)}{2} \\ &= \frac{n^2 - 3n - 4}{2} < \frac{n^2 - 3n + 2}{2} = e(K_{n-1}). \end{aligned}$$

Thus, $e(G) < e((G - V_1) \cup K_{n-1})$, which contradicts the assumption $G \in Ex(p; T_n''')$.

Now assume $t \geq 3$. If $|\Gamma(v_1) \cap \Gamma_2(v_0)| = t$, then $u_i v_j \notin E(G)$ for any $i \in \{1, 2, \dots, t\}$ and $j \in \{2, 4, \dots, n - 4\}$. We see that

$$\begin{aligned} e(G) - e(G - V_1) &\leq d(v_1) + d(u_1) + d(u_2) + e(G[v_0, v_2, \dots, v_{n-4}]) \\ &\leq n - 4 + n - 4 + n - 4 + \frac{(n - 4)(n - 5)}{2} = \frac{n^2 - 3n - 4}{2} \\ &< \frac{(n - 1)(n - 2)}{2} = e(K_{n-1}) \end{aligned}$$

and so $e(G) < e((G - V_1) \cup K_{n-1})$. This contradicts the assumption $G \in Ex(p; T_n''')$. If $|\Gamma(v_1) \cap \Gamma_2(v_0)| \geq 1$ and $|\Gamma(v_2) \cap \Gamma_2(v_0)| \geq 1$, then $u_1 v_1, u_2 v_2 \in E(G)$, $u_i v_j \notin E(G)$ for any $i \in \{3, 4, \dots, t\}$ and $j \in \{3, 4, \dots, n - 4\}$. Thus,

$$\begin{aligned} e(G) - e(G - V_1) &\leq d(v_1) + d(v_2) + d(u_1) + d(u_2) + e(G[v_0, v_3, \dots, v_{n-4}]) \\ &\leq n - 4 + n - 4 + n - 4 + n - 4 + \frac{(n - 5)(n - 6)}{2} \end{aligned}$$

$$= \frac{n^2 - 3n - 2}{2} < \frac{(n-1)(n-2)}{2} = e(K_{n-1}).$$

This contradicts the fact $G \in Ex(p; T_n''')$. Hence $\Delta(G) = n - 5$ as claimed.

Theorem 5.1. *Let $p, n \in \mathbb{N}, p \geq n \geq 10, p = k(n-1) + r, k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. Then*

$$ex(p; T_n''') = \frac{(n-2)p - r(n-1-r)}{2} + \max \left\{ 0, \left\lceil \frac{r(n-4-r) - 3(n-1)}{2} \right\rceil \right\}.$$

Proof. This is immediate from Lemmas 2.10 and 5.2.

References

- [1] R.J. Faudree and R.H. Schelp, Path Ramsey numbers in multicolorings, J. Combin. Theory Ser. B **19**(1975), 150-160.
- [2] A.F. Sidorenko, Asymptotic solution for a new class of forbidden r -graphs, Combinatorica **9**(1989), 207-215.
- [3] Z.H. Sun and L.L.Wang, Turán's problem for trees, J. Combin. Number Theory **3**(2011), 51-69.
- [4] Z.H. Sun, L.L.Wang and Y.L. Wu, Turán's problem and Ramsey numbers for trees, arXiv:1110.2725.
- [5] M. Woźniak, On the Erdős-Sós conjecture, J. Graph Theory **21**(1996), 229-234.